Robust feedback switching control

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Switching control

• Switching control : sequence of *interventions* $(\iota_n)_n$ that occur at *random times* $(\tau_n)_n$ due to switching costs, and naturally arises in investment problems with fixed transaction costs or in real options.

• *Standard approach :*

- **open-loop** ($\neq$ closed-loop) control
- give the evolution for the controlled state process, with *assigned* drift and diffusion coefficients.

• In practice, the coefficients are obtained through estimation procedures and are unlikely to coincide with the *real* coefficients.

• *Robust approach :* switching control problem *robust* to a misspecification of the model for the controlled state process.

Robust/Game formulation

• We formulate the problem as a **game :** switcher *vs* nature (model uncertainty).

▶ We consider the *two-step optimization* problem

$$\sup_{\alpha}\left(\inf_{\upsilon} J(\alpha, \upsilon)\right).$$

• What definition for the switching control α and for υ ?

Feedback formulation

- Elliott-Kalton formulation (Fleming-Souganidis 89) :
  - α **non-anticipative strategy** and υ *open-loop control*, i.e. the switcher knows the current and past choices made by nature
  - In practice, the switcher only knows the evolution of the state process.

▶ Feedback formulation
  - α *feedback switching control* (**closed-loop control**) $\Longrightarrow$ *feedback formulation* of the switching control problem.
  - υ *open-loop control* (nature is aware of the all information at disposal) $\leftrightarrow$ Knightian uncertainty

$\rightarrow$ zero-sum control/control game but not symmetric

Outline

Outline

1. Model setup
2. Stochastic Perron's method and Hamilton-Jacobi-Bellman-Isaacs equation
3. Ergodicity

Robust feedback switching system

- Fixed $(\Omega, \mathcal{F}, \mathbb{P})$, $T > 0$, and W a d-dimensional Brownian motion.

For any $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$, consider the system on $\mathbb{R}^d \times \mathbb{I}_m$, with $\mathbb{I}_m = \{1, \ldots, m\}$ the set of regimes :

$$\begin{cases} X_t = x + \int_s^t b(X_r, I_r, \upsilon_r)dr + \int_s^t \sigma(X_r, I_r, \upsilon_r)dW_r, & s \leqslant t \leqslant T, \\ I_t = i1_{\{s \leqslant t < \tau_0(X_\cdot, I_{\cdot -})\}} \\ \qquad + \sum_{n \in \mathbb{N}} \iota_n(X_\cdot, I_{\cdot -})1_{\{\tau_n(X_\cdot, I_{\cdot -}) \leqslant t < \tau_{n+1}(X_\cdot, I_{\cdot -})\}}, & s \leqslant t < T, \\ I_{s^-} = I_s, \; I_T = I_{T^-}. \end{cases}$$

▶ $\upsilon : [s, T] \times \Omega \to U$ is an open-loop control adapted to a filtration $\mathbb{F}^s = (\mathcal{F}_t^s)_{t \geqslant s}$ satisfying the usual conditions.

- U compact metric space.

$\mathcal{U}_{s,s}$: class of all open-loop controls starting at s.

Feedback switching controls

- $\mathscr{L}([s,T];\mathbb{I}_m)$ space of **càglàd** paths valued in $\mathbb{I}_m$.
- $\mathbb{B}^s = (\mathcal{B}^s_t)_{t\in[s,T]}$ natural filtration of $C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m)$.
- $\mathcal{T}^s$ family of all $\mathbb{B}^s$-stopping times valued in $[s,T]$.

▶ Feedback switching control $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}}$ where :

- Switching times : $\tau_n \in \mathcal{T}^s$ and

$$s \leqslant \tau_0 \leqslant \cdots \leqslant \tau_n \leqslant \cdots \leqslant T.$$

- Interventions : $\iota_n \colon C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m) \to \mathbb{I}_m$ is $\mathcal{B}^s_{\tau_n}$-measurable, for any $n \in \mathbb{N}$.

▶ $\mathcal{A}_{s,s}$: class of all feedback switching controls starting at s.

Existence and uniqueness result

(H1) b and σ jointly continuous on $\mathbb{R}^d \times \mathbb{I}_m \times U$ and

$$|b(x,i,u) - b(x',i,u)| + \|\sigma(x,i,u) - \sigma(x',i,u)\| \leqslant L|x - x'|.$$

Proposition

Let **(H1)** *hold. Then, for every* $(s,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m$, $\alpha \in \mathcal{A}_{s,s}$, $\upsilon \in \mathcal{U}_{s,s}$, *there exists a unique* $\mathbb{F}^s$*-adapted solution* $(X_t^{s,x,i;\alpha,u}, I_t^{s,x,i;\alpha,u})_{t\in[s,T]}$ *to the feedback system, satisfying :*

- *Every path of* $(X_\cdot^{s,x,i;\alpha,\upsilon}, I_{\cdot-}^{s,x,i;\alpha,\upsilon})$ *belongs to* $C([s,T];\mathbb{R}^d) \times \mathscr{L}([s,T];\mathbb{I}_m)$.
- *For any* $p \geqslant 1$ *there exists a positive constant* $C_{p,T}$ *such that*

$$\mathbb{E}\Big[\sup_{t\in[s,T]} |X_t^{s,x,i;\alpha,\upsilon}|^p\Big] \leqslant C_{p,T}(1 + |x|^p).$$

Value function of robust switching control problem

Feedback control/open-loop control game :

$$V(s,x,i) := \sup_{\alpha\in\mathcal{A}_{s,s}} \inf_{\upsilon\in\mathcal{U}_{s,s}} J(s,x,i;\alpha,\upsilon), \quad \forall (s,x,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{I}_m,$$

with

$$\begin{aligned} J(s,x,i;\alpha,\upsilon) \ := \ \mathbb{E}\Big[& \int_s^T f(X_r^{s,x,i;\alpha,\upsilon}, I_r^{s,x,i;\alpha,\upsilon}, \upsilon_r)dr \\ & + \ g(X_T^{s,x,i;\alpha,\upsilon}, I_T^{s,x,i;\alpha,\upsilon}) \\ & - \sum_{n\in\mathbb{N}} c(X_{\tau_n}^{s,x,i;\alpha,\upsilon}, I_{\tau_n^-}^{s,x,i;\alpha,\upsilon}, I_{\tau_n}^{s,x,i;\alpha,\upsilon})1_{\{s\leqslant\tau_n<T\}}\Big], \end{aligned}$$

where τ^n stands for $\tau^n(X_\cdot^{s,x,i;\alpha,\upsilon}, I_{\cdot-}^{s,x,i;\alpha,\upsilon})$.

Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation

$$\begin{cases} \min\Big\{ -\dfrac{\partial V}{\partial t}(s,x,i) - \inf_{u\in U}\big[\mathcal{L}^{i,u}V(s,x,i) + f(x,i,u)\big], \\ V(s,x,i) - \max_{j\neq i}\big[V(s,x,j) - c(x,i,j)\big]\Big\} \;=\; 0, \quad [0,T)\times\mathbb{R}^d\times\mathbb{I}_m \\ V(T,x,i) \;=\; g(x,i), \quad (x,i)\in\mathbb{R}^d\times\mathbb{I}_m, \end{cases}$$

where

$$\mathcal{L}^{i,u}V(s,x,i) \;=\; b(x,i,u).D_xV(s,x,i) + \frac{1}{2}\mathrm{tr}\big[\sigma\sigma^{\intercal}(x,i,u)D_x^2V(s,x,i)\big].$$

- First aim : prove that V is a viscosity solution to the *dynamic programming HJBI equation* :
  - by **stochastic Perron** method : **avoiding the direct proof of Dynamic Programming Principle** (DPP)

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Stochastic Perron : main idea

Developed in a series of papers by B. and Sirbu

• Define **stochastic sub and super-solutions** as functions that satisfy (roughly) half of the DPP

▶ with these definitions, sub and super-solutions envelope the value function

• Consider sup of sub-solutions and inf of super-solutions (Perron) :

$$v^- \ := \ \text{sup of sub-solutions} \ \leqslant \ V \ \leqslant \ v^+ \ := \ \text{inf of super-solutions}$$

▶ Show that v^- is a viscosity super-solution and v^+ is a viscosity sub-solution.

• Comparison principle $\rightarrow$

$$v^- \ = \ V \ = \ v^+ \qquad \text{is the unique continuous viscosity solution.}$$

and (as a byproduct) V satisfies the DPP

Some comments

• Stochastic semi-solutions have to be carefully defined (depending on the control problem) $\rightarrow$ constructive proof for the existence of a viscosity solution comparing with the value function

- linear, control, optimal stopping problems (Bayraktar-Sirbu, 12, 13, 14.)

Stochastic semisolutions

Definition (Stochastic subsolutions $\mathcal{V}^-$)

v stochastic subsolution to the HJBI equation if :

- v is continuous, $v(T,x,i) \leqslant g(x,i)$ for any $(x,i) \in \mathbb{R}^d \times \mathbb{I}_m$, and $\sup_{(s,x,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{I}_m} \frac{|v(s,x,i)|}{1+|x|^q} < \infty$, for some $q \geqslant 1$.
- Half-DPP property. For any $s \in [0,T]$ and $\tau, \rho \in \mathcal{T}^s$ with $\tau \leqslant \rho \leqslant T$, there exists $\widetilde{\alpha} = (\widetilde{\tau}_n, \widetilde{\iota}_n)_{n\in\mathbb{N}} \in \mathcal{A}_{s,\tau^+}$ such that, for any $\alpha = (\tau_n, \iota_n)_{n\in\mathbb{N}} \in \mathcal{A}_{s,s}$, $\upsilon \in \mathcal{U}_{s,s}$, and $(x,i) \in \mathbb{R}^d \times \mathbb{I}_m$, we have

$$v(\tau', X_{\tau'}, I_{\tau'}) \leqslant \mathbb{E}\Bigg[\int_{\tau'}^{\rho'} f(X_t, I_t, \upsilon_t)dt + v(\rho', X_{\rho'}, I_{\rho'}) - \sum_{n\in\mathbb{N}} c(X_{\widetilde{\tau}'_n}, I_{(\widetilde{\tau}'_n)^-}, I_{\widetilde{\tau}'_n}) 1_{\{\tau' \leqslant \widetilde{\tau}'_n < \rho'\}} \Bigg| \mathcal{F}^s_{\tau'}\Bigg]$$

with the shorthands $X = X^{s,x,i;\alpha\otimes_\tau\widetilde{\alpha},\upsilon}$, $I = I^{s,x,i;\alpha\otimes_\tau\widetilde{\alpha},\upsilon}$.

▶ The set of *stochastic supersolutions* $\mathcal{V}^+$ is defined similarly.

Stochastic Perron's method : assumptions

(H2)

(i) g, f, c are jointly continuous on their domains.

(ii) c is nonnegative.

(iii) g, f, c satisfy the polynomial growth condition :

$$|g(x,i)| + |f(x,i,u)| + |c(x,i,j)| \quad \leqslant \quad M(1+|x|^p),$$

$\forall\, x \in \mathbb{R}^d$, $i, j \in \mathbb{I}_m$, $u \in U$, for some positive constants M and $p \geqslant 1$.

(iv) g satisfies

$$g(x,i) \quad \geqslant \quad \max_{j \neq i} \left[g(x,j) - c(x,i,j) \right],$$

for any $x \in \mathbb{R}^d$ and $i \in \mathbb{I}_m$.

Stochastic Perron's method

Proposition

Let Assumptions **(H1)** *and* **(H2)** *hold.*

(i) $\mathcal{V}^- \neq \emptyset$ *and* $\mathcal{V}^+ \neq \emptyset$.

(ii) $\sup_{v \in \mathcal{V}^-} v =: v^- \leqslant V \leqslant v^+ := \inf_{v \in \mathcal{V}^+} v$.

(iii) *If* $v^1, v^2 \in \mathcal{V}^-$ *then* $v := v^1 \vee v^2 \in \mathcal{V}^-$. *Moreover, there exists a nondecreasing sequence* $(v_n)_n \subset \mathcal{V}^-$ *such that* $v_n \nearrow v^-$.

(iv) *If* $v^1, v^2 \in \mathcal{V}^+$ *then* $v := v^1 \wedge v^2 \in \mathcal{V}^+$. *Moreover, there exists a nonincreasing sequence* $(v_n)_n \subset \mathcal{V}^+$ *such that* $v_n \searrow v^+$.

Theorem [Stochastic Perron's method]

Let Assumptions **(H1)** and **(H2)** hold. Then, v^- is a viscosity supersolution to the HJBI equation and v^+ is a viscosity subsolution to the HJB equation.

Comparison principle

(H3) c satisfies the no free loop property : for any sequence of indices $i_1, \ldots, i_k \in \mathbb{I}_m$, with $k \in \mathbb{N}\backslash\{0,1,2\}$, $i_1 = i_k$, and $\text{card}\{i_1, \ldots, i_k\} = k-1$, we have

$$c(x, i_1, i_2) + c(x, i_2, i_3) + \cdots + c(x, i_{k-1}, i_k) + c(x, i_k, i_1) > 0.$$

We also assume : $c(x, i, i) = 0$, $\forall\,(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$.

Theorem [Comparison principle]

Let Assumptions **(H1)**, **(H2)**, **(H3)** hold and consider a viscosity subsolution u (resp. supersolution v) to the HJB equation. Suppose that, for some $q \geqslant 1$,

$$\sup_{(t,x,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{I}_m} \frac{|u(t,x,i)| + |v(t,x,i)|}{1+|x|^q} < \infty.$$

Then, $u \leqslant v$ on $[0, T] \times \mathbb{R}^d \times \mathbb{I}_m$.

Dynamic programming and viscosity properties

Theorem

Let Assumptions **(H1)**, **(H2)**, **(H3)** hold. Then, the value function V is the unique viscosity solution to the HJB equation and satisfies the dynamic programming principle : for any $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ and $\rho \in \mathcal{T}^s$,

$$V(s,x,i) = \sup_{\alpha \in \mathcal{A}_{s,s}} \inf_{\upsilon \in \mathcal{U}_{s,s}} \mathbb{E}\Big[\int_s^{\rho'} f(X_t, I_t, \upsilon_t)dt + V(\rho', X_{\rho'}, I_{\rho'}) - \sum_{n \in \mathbb{N}} c(X_{\tau_n'}, I_{(\tau_n')-}, I_{\tau_n'}) 1_{\{s \leqslant \tau_n' < \rho'\}} \Big],$$

with the shorthands $X = X^{s,x,i;\alpha,\upsilon}$, $I = I^{s,x,i;\alpha,\upsilon}$, $\rho' = \rho(X_\cdot, I_{\cdot-})$, $\tau_n' = \tau_n(X_\cdot, I_{\cdot-})$, and $\upsilon_t' = \upsilon(t, X_\cdot, I_{\cdot-})$.

Comparison with the Elliott-Kalton formulation

- In general $V \leqslant V^{\text{Kalton}}$.
- And if comparison principle holds we have equality. BUT, intrinsically they are different problems and two formulations lead to two different solutions of the variational HJB.
- We have an example with $c \equiv 0$ (hence the no-free loop is violated), where each formulation leads to different solutions of the variational HJB : $V < V^{\text{Kalton}}$
- Since $c \equiv 0$ this actually can be reformulated as a classical zero-sum game
  - V is the solution to the lower Isaacs equation.
  - V^{Kalton} is the solution to the upper Isaacs equation.
  - The Isaacs condition does not hold.

Outline

Problem

Forward parabolic system of variational inequalities :

$$\begin{cases} \min \Big\{ \dfrac{\partial V}{\partial T} - \inf_{u \in U} \big[\mathcal{L}^{i,u} V + f(x,i,u) \big], \\ V(T,x,i) - \max_{j \neq i} \big[V(T,x,j) - c(x,i,j) \big] \Big\} \; = \; 0, \; (0,\infty) \times \mathbb{R}^d \times \mathbb{I}_m \\ V(0,x,i) \; = \; g(x,i), \qquad (x,i) \in \mathbb{R}^d \times \mathbb{I}_m \end{cases}$$

- Long time asymptotics of $V(T,\cdot,\cdot)$ as $T \to \infty$:

- Stationary solution of robust feedback switching control
- Literature on ergodic stochastic control : Switching (Lions, Perthame (86), Menaldi, Perthame, Robin (90)), Stochastic control (Bensoussan, Frehse (92) ; Arisawa, P.L. Lions (98)).
- More recently Lions' College de France lectures, Ichihara, Ishii (08), Fuhrman, Hu and Tessitore (09), Ichihara (2012), Robertson, Xing (15)...
- **under non degeneracy condition and/or regularity of value function and very few on games !**

Some heuristics and principles

- We expect to prove (under suitable conditions) that

$$\frac{V(T,x,i)}{T} \quad \rightarrow \quad \lambda \ (\textbf{const. independent of} \ \ \text{x,i}) \quad \text{as } T \rightarrow \infty.$$

- **Tauberian Meta theorem** : ergodic $\sim$ infinite horizon with vanishing discount factor, i.e.

$$\lim_{T\rightarrow\infty} \frac{V(T,.)}{T} \quad = \quad \lim_{\beta \rightarrow 0} \beta V^{\beta}$$

where

$$V^{\beta}(x,i) \quad = \quad \sup_{\alpha \in \mathcal{A}_{0,0}} \inf_{\upsilon \in \mathcal{U}_{0,0}} \mathbb{E}\Big[\int_0^{\infty} e^{-\beta t} f(X_t^{x,i;\alpha,\upsilon}, I_t^{x,i;\alpha,\upsilon}, \upsilon_t) dt - \sum_{n \in \mathbb{N}} e^{-\beta \tau_n} c(X_{\tau_n}^{x,i;\alpha,u}, I_{\tau_n^-}^{x,i;\alpha,\upsilon}, I_{\tau_n}^{x,i;\alpha,\upsilon}) 1_{\{\tau_n < \infty\}} \Big]$$

$\leftrightarrow$ Elliptic system of variational inequalities :

$$\min\Big\{ \beta V^{\beta} - \inf_{u \in U} \big[\mathcal{L}^{i,u} V^{\beta} + f(x,i,u) \big]; V^{\beta}(x,i) - \max_{j \neq i} \big[V^{\beta}(x,j) - c(x,i,j) \big] \Big\} \quad = \quad 0.$$

Ergodic system of variational inequalities

- Formally, by setting $V(T,x,i) \sim \lambda T + \phi(x,i)$ as $T \to \infty$, we get the ergodic HJBI equation :

$$\min \left\{ \lambda - \inf_{u \in U} \left[\mathcal{L}^{i,u}\phi + f(x,i,u) \right], \phi(x,i) - \max_{j \neq i} \left[\phi(x,j) - c(x,i,j) \right] \right\} = 0.$$

▶ The pair (λ, ϕ) is the unknown.

- Aim :
  - Prove existence (and uniqueness) of a solution to the ergodic HJBI
  - Show :

$$\lim_{T \to \infty} \frac{V(T,x,i)}{T} = \lambda = \lim_{\beta \to 0} \beta V^{\beta}(x,i).$$

Main issues for asymptotic analysis

- Prove equicontinuity of the family $(V^\beta)_\beta$: for all $\beta > 0$,

$$\begin{aligned} |V^\beta(x,i) - V^\beta(x',i)| &\leqslant C|x-x'|, \\ \beta|V^\beta(x,i)| &\leqslant C(1+|x|), \quad \forall\,(x,i). \end{aligned}$$

  - by PDE methods from the elliptic HJBI system ?
  - from the robust feedback switching control representation, which would rely on an estimate of the form :

$$\sup_{\alpha\in\mathcal{A}_{0,0},\upsilon\in\mathcal{U}_{0,0}} \mathbb{E}\left|X_t^{x,i;\alpha,\upsilon} - X_t^{x',i;\alpha,\upsilon}\right| \leqslant C_t|x-x'|, \quad \forall x, x', i.$$

  Not clear due to the feedback form of the switching control !

Randomization of the control

Following idea of Kharroubi and Pham (13) :

$$\begin{cases} X_t = x + \int_0^t b(X_s, I_s, \Gamma_s)ds + \int_0^t \sigma(X_s, I_s, \Gamma_s)dW_s, \\ I_t = i + \int_0^t \int_{\mathbb{I}_m} (j - I_{s-})\pi(ds, dj), \\ \Gamma_t = u + \int_0^t \int_U (u' - \Gamma_{s-})\mu(ds, du'), \end{cases}$$

• π Poisson random measure on $\mathbb{R}_+ \times \mathbb{I}_m$, μ Poisson random measure on $\mathbb{R}_+ \times U$. W, π, and μ are *independent*.

▶ $(X^{x,i,u}, I^i, \Gamma^u)$ exogenous (uncontrolled) Markov process

Change of equivalent probability measures

Control of intensity measures :

• Ξ (resp. $\mathcal{V}$) class of *essentially bounded predictable* maps $\xi\colon [0,\infty)\times\Omega\times\mathbb{I}_m \to (0,\infty)$ (resp. $\nu\colon [0,\infty)\times\Omega\times U \to [1,\infty)$)

$$\frac{d\mathbb{P}^{\xi,\nu}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \mathcal{E}_T\Big(\int_0^{\cdot}\int_{\mathbb{I}_m}(\xi_t(j)-1)\widetilde{\pi}(dt,dj)\Big)\cdot\mathcal{E}_T\Big(\int_0^{\cdot}\int_U(\nu_t(u')-1)\widetilde{\mu}(dt,du')\Big)$$

▶ **Under $\mathbb{P}^{\xi,\nu}$:**

- W remains a Brownian motion.
- $\mathbb{P}$-compensator $\vartheta_\pi(di)dt$ of $\pi \longrightarrow \xi_t(i)\vartheta_\pi(di)dt$.
- $\mathbb{P}$-compensator $\vartheta_\mu(du)dt$ of $\mu \longrightarrow \nu_t(u)\vartheta_\mu(du)dt$.

$\rightarrow$ Easy to derive moment and Lipschitz estimates on $X^{x,i,u}$ under $\mathbb{P}^{\xi,\nu}$!

Dual robust switching control

$$v^\beta(x,i,u) \;=\; \sup_{\xi\in\Xi}\inf_{\nu\in\mathcal{V}} \mathbb{E}^{\xi,\nu}\Big[\int_0^\infty e^{-\beta t} f(X_t^{x,i,u}, I_t^i, \Gamma_t^u)dt - \int_0^\infty\int_{\mathbb{I}_m} e^{-\beta t} c(X_{t^-}^{x,i,u}, I_{t^-}^i, j)\pi(dt,dj)\Big],$$

for all $(x,i,u) \in \mathbb{R}^d \times \mathbb{I}_m \times U$.

- The dual problem is a **symmetric game** : *control vs control*.

Theorem

For any $\beta > 0$ and $(x,i) \in \mathbb{R}^d \times \mathbb{I}_m$,

$$v^\beta(x,i,u) \;=\; v^\beta(x,i,u'), \qquad \forall\, u,u' \in U$$

and

$$V^\beta(x,i) \;=\; v^\beta(x,i,u), \qquad \forall\,(x,i) \in \mathbb{R}^d \times \mathbb{I}_m,$$

for any $u \in U$.

BSDE Representation of the weak control problem

$$
\begin{aligned}
Y_t^{\beta,n} \quad = \quad & Y_T^{\beta,n} - \beta \int_t^T Y_s^{\beta,n} ds + \int_t^T f(X_s^{x,i,u}, I_s^i, \Gamma_s^u) ds - \sum_{j=1}^m \int_t^T L_s^{\beta,n}(j) ds \\
& + n \sum_{j=1}^m \int_t^T \left[L_s^{\beta,n}(j) - c(X_s^{x,i,u}, I_{s-}^i, j) \right]^+ ds - \left(K_T^{\beta,n} - K_t^{\beta,n} \right) \qquad (1) \\
& - \int_t^T Z_s^{\beta,n} dW_s - \int_t^T \int_{\mathbb{I}_m} L_s^{\beta,n}(j) \widetilde{\pi}(ds, dj) - \int_t^T \int_U R_s^{\beta,n}(u') \widetilde{\mu}(ds, du'),
\end{aligned}
$$

for any $0 \leqslant t \leqslant T$, $T \in [0, \infty)$, and

$$
R_t^{\beta,n}(u') \quad \geqslant \quad 0, \qquad d\mathbb{P} \otimes dt \otimes \vartheta_\mu(du')\text{-a.e.} \qquad (2)
$$

Ergodicity under dissipativity condition

- Dissipativity condition (DC) : for all $x, x' \in \mathbb{R}^d$, $i \in \mathbb{I}_m$, $u \in U$,

$$
\begin{aligned}
& (x - x').(b(x,i,u) - b(x',i,u)) \ + \ \frac{1}{2}\|\sigma(x,i,u) - \sigma(x',i,u)\|^2 \\
\leqslant \quad & -\gamma\,|x - x'|^2
\end{aligned}
$$

for some constant $\gamma > 0$.

$\Longrightarrow$

$$
\begin{aligned}
\sup_{\xi,\nu} \mathbb{E}^{\xi,\nu}[|X_t^{x,i,u} - X_t^{x',i,u}|^2] \quad & \leqslant \quad e^{-2\gamma t}|x - x'|^2 \\
\sup_{t \geqslant 0} \sup_{\xi,\nu} \mathbb{E}^{\xi,\nu}|X_t^{x,i,u}| \quad & \leqslant \quad C(1 + |x|).
\end{aligned}
$$

Main steps of proof for existence to ergodic system

- **Equicontinuity** :

$$
\begin{aligned}
&|V^\beta(x,i) - V^\beta(x',i)| \\
&\leqslant \sup_{\xi\in\Xi,\,\nu\in\mathcal{V}} \mathbb{E}^{\xi,\nu}\left[\int_0^\infty e^{-\beta t}\left|f(X_t^{x,i,u}, I_t^i, \Gamma_t^u) - f(X_t^{x',i,u}, I_t^i, \Gamma_t^u)\right| dt\right] \\
&\leqslant L|x-x'|\int_0^\infty e^{-(\beta+\gamma)t}dt \;=\; \frac{L}{\beta+\gamma}|x-x'| \;\leqslant\; \frac{L}{\gamma}|x-x'|.
\end{aligned}
$$

- **Convergence of V^β.** Define

$$
\lambda_i^\beta := \beta V^\beta(0,i), \qquad \phi^\beta(x,i) := V^\beta(x,i) - V^\beta(0,i_0),
$$

By Bolzano-Weierstrass and Ascoli-Arzelà theorems, we can find a sequence $(\beta_k)_{k\in\mathbb{N}}$, with $\beta_k \searrow 0^+$, such that

$$
\lambda_i^{\beta_k} \xrightarrow{k\to\infty} \lambda_i, \qquad \phi^{\beta_k}(\cdot,i) \underset{\text{in } C(\mathbb{R}^d)}{\xrightarrow{k\to\infty}} \phi(\cdot,i).
$$

▶ $\lambda := \lambda_i$ does not depend on $i \in \mathbb{I}_m$.
Finally, stability results of viscosity solutions $\Longrightarrow$ (λ,ϕ) is a viscosity solution to the ergodic system.

A simple argument for large time convergence

Let (λ, ϕ) be a solution to the ergodic HJBI :

- ϕ is the unique viscosity solution to the parabolic HJBI equation with unknown ψ and terminal condition ϕ :

$$\begin{cases} \min\Big\{-\frac{\partial \psi}{\partial t}(t,x,i) - \inf_{u\in U}\left[\mathcal{L}^{i,u}\psi(t,x,i) + f(x,i,u) - \lambda\right], \\ \psi(t,x,i) - \max_{j\neq i}\left[\psi(t,x,j) - c(x,i,j)\right]\Big\} \;=\; 0, \qquad (t,x,i)\in[0,T)\times\mathbb{R}^d\times\mathbb{I}_m, \\ \psi(T,x,i) \;=\; \phi(x,i), \qquad (x,i)\in\mathbb{R}^d\times\mathbb{I}_m. \end{cases}$$

- For any $T > 0$, $\phi(x,i)$ admits the dual game representation :

$$\begin{aligned} \phi(x,i) \quad = \quad \sup_{\xi\in\Xi}\inf_{\nu\in\mathcal{V}} \mathbb{E}^{\xi,\nu}\Big[\int_0^T \big(f(X_t^{x,i,u}, I_t^i, \Gamma_t^u) - \lambda\big)dt + \phi(X_T^{x,i,u}, I_T^i) \\ - \int_0^T\int_{\mathbb{I}_m} e^{-\beta t} c(X_{t^-}^{x,i,u}, I_{t-}^i, j)\pi(dt,dj)\Big] \end{aligned}$$

Large time convergence (Ctd and end)

From the dual game representation for $V(T,.)$:

$$
\begin{aligned}
& \left|V(T,x,i) - \lambda T - \phi(x,i)\right| \\
\leqslant \quad & \sup_{\xi \in \Xi, \nu \in \mathcal{V}} \mathbb{E}^{\xi,\nu}\Big[\left|g(X_T^{x,i}, I_T^i)\right| + \max_j \left|\phi(X_T^{x,}, j)\right|\Big] \\
\leqslant \quad & C(1 + |x|^2),
\end{aligned}
$$

from growth condition of g, ϕ, and estimate of X under dissipativity condition.
$\Longrightarrow$

$$
\frac{V(T,x,i)}{T} \quad \to \quad \lambda, \quad \text{as} \quad T \to \infty.
$$

Remark. This probabilistic argument does not require any non degeneracy condition on σ, hence any regularity on value functions.

Concluding remarks

- Robust (model uncertainty) feedback switching control :
  - Non symmetric zero-sum control/control game
  - $\neq$ Elliott-Kalton game formulation
- Stochastic Perron method
  - HJBI equation and DPP
- Ergodicity of HJBI
  - Randomization method $\rightarrow$ dual symmetric (open loop) control/control game representation
  - No non-degeneracy condition

Main References

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Thank you !