Supply and Shorting in Speculative Markets

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Outline

Static Model

Consider

- Agents $i \in \{1, 2, \dots, n\}$
- Using distributions Q_i for the state $X(t)$
- Trading an asset with a single payoff $f(X(T))$ at time T
- The asset cannot be shorted and is in supply $s > 0$

Static case: Suppose the agents trade only once, at time $t = 0$

Equilibrium:

- Determine an equilibrium price p for f and portfolios $q_i \in \mathbb{R}_+$
- such that q_i maximizes $q(E_i[f(X(T))] - p)$ over $q \geq 0$, for all i
- and the market clears: $\sum_i q_i = s$

Static Equilibrium

Solution: The most optimistic agent determines the price (Miller '77),

$$p = \max_i E_i[f(X(T))]$$

- Let $i_* \in \{1, 2, \ldots, n\}$ be the maximizer
- With portfolios $q_{i_*} = s$ and $q_i = 0$ for $i \neq i_*$, this in an equilibrium
- It is unique (modulo having several maximizers)

Note:

- At price p, the optimist is invariant and will accept any portfolio
- All other agents want to have $q_i = 0$
- Price not affected by supply

Preview: The Resale Option (Harrison and Kreps '78)

- When there are several trading dates, the relatively most optimistic agent depends on date and state
- Option to resell the asset to another agent at a later time
- Adds to the static price: speculative bubble

Scheinkman and Xiong '03, '04

A Continuous-Time Model

Asset can be traded on $[0, T]$.

Agents:

- Risk-neutral agents $i \in \{1, \ldots, n\}$ using models Q_i
- Here: agent i uses a local vol model Q_i for X,

$$dX(t) = \sigma_i(t, X(t))\, dW_i(t), \quad X(0) = x$$

Equilibrium:

- Find a price process $P(t)$ with $P(T) = f(X(T))$ Q_i-a.s.
- Agents choose portfolio processes Φ
- such as to optimize expected P&L: $E_i[\int_0^T \Phi(t)\, dP(t)]$
- Market clearing $\sum_i \Phi_i(t) = s$

Existence

Theorem: There exists a unique equilibrium price $P(t) = v(t, X(t))$, and v is the solution of

$$v_t(t,x) + \sup_{i \in \{1,\dots,n\}} \frac{1}{2}\sigma_i^2(t,x) v_{xx}(t,x) = 0, \quad v(T,\cdot) = f.$$

The optimal portfolios $\Phi_i(t) = \phi_i(t, X(t))$ are given by

$$\phi_i(t,x) = \begin{cases} s, & \text{if } i \text{ is the maximizer at } (t,x) \\ 0, & \text{else} \end{cases}$$

- Derivative held by the locally most optimistic agent at any time
- Agents trade as this role changes

Control Problem and Speculative Bubble

- v is also characterized as the value function

$$v(t,x) = \sup_{\theta \in \Theta} E[f(X_\theta^{t,x}(T))]$$

  - Θ is the set of $\{1,\dots,n\}$-valued, progressive processes
  - $X_\theta^{t,x}(r)$, $r \in [t,T]$ is the solution of

$$dX(r) = \sigma_{\theta(r)}(r, X(r))\, dW(t), \quad X(t) = x.$$

Bubble:

- The control problem (or comparison) shows that

$$P(0) \geq \max_i E_i[f(X(T))]$$

- Thus, the price exceeds the static equilibrium
- This "speculative bubble" can be attributed to the resale option

Remarks

Note:

- Price is again independent of supply
- No-shorting was essential

Comparison with UVM:

- v is the uncertain volatility (UV) or G-expectation price corresponding to the interval

$$[\underline{\sigma}, \overline{\sigma}] = \left[\inf_i \sigma_i(t,x), \sup_i \sigma_i(t,x)\right]$$

$\rightarrow$ In our model, the UV price arises as an equilibrium price of risk-neutral agents, instead of a superhedging price

Heston Example

Consider $n = 2$ agents with stochastic volatility models $X = (S, Y)$, where

$$dS(t) = \alpha(Y(t))\,dW(t), \quad S(0) = s,$$
$$dY(t) = \lambda_i(\bar{Y} - Y(t))\,dt + \beta(Y(t))\,dW'(t), \quad Y(0) = y$$

- Agents disagree on the (unobservable) speed of mean reversion λ_i
- $\lambda_1 > \lambda_2$
- For a convex payoff $f(S(T))$, the optimal portfolios are

$$\phi_1(t, y) = \begin{cases} 1, & y \leq \bar{Y}, \\ 0, & y > \bar{Y} \end{cases}$$

  and $\phi_2 = 1 - \phi_1$
- Optimism corresponds to expecting an increase of volatility

Outline

Model

Supply:

- Supply should diminish price, not reflected in the model of Part I
- Need (risk) aversion against large positions

Add Cost-of-Carry: For holding a position $y = \Phi(t)$ at time t, agents must pay an instantaneous cost

$$c(y) = \begin{cases} \frac{1}{2\alpha_+} y^2, & y \geq 0 \\ \infty, & y < 0 \end{cases}$$

Equilibrium: Agents optimize expected P&L − cost:

$$E_i \left[\int_0^T \Phi(t)\, dP(t) - \int_0^T c(\Phi(t))\, dt \right]$$

Existence

Theorem: • There exists a unique equilibrium price $P(t) = v(t, X(t))$, and v is the solution of

$$v_t + \sup_{\emptyset \neq J \subseteq \{1,\dots,n\}} \left\{ \frac{1}{|J|} \sum_{i \in J} \frac{1}{2} \sigma_i^2 v_{xx} - \frac{s}{|J| \alpha_+} \right\} = 0$$

• The optimal portfolios $\Phi_i(t) = \phi_i(t, X(t))$ are unique and given by

$$\phi_i(t,x) = \left\{ \alpha_+ \mathcal{L}^i v(t,x) \right\}^+$$

where $\mathcal{L}^i v(t,x) = \partial_t v(t,x) + \frac{1}{2} \sigma_i^2 \partial_{xx} v(t,x)$

Supply: enters as a running cost, $\kappa = \frac{s}{|J| \alpha_+}$

Delay Effect

- Again, one can consider a static version of the equilibrium: price is

$$p = \max_{\emptyset \neq J \subseteq \{1,\ldots,n\}} \left(\frac{1}{|J|} \sum_{i \in J} E_i[f(X(T))] - \frac{sT}{|J|\alpha_+} \right).$$

- The resale option is still present and increases the dynamic price

- Novel: Delay Effect
- If many agents expect to increase positions over time, they may anticipate the increase in the static case
- The resulting demand pressure raises the static price

- This effect may dominate, causing a "negative bubble"

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Outline

Short-Selling

- In securities markets, shorting is often possible, though at a cost
- Not modeled in the existing literature

Asymmetric Cost-of-Carry:

- For holding a position $y = \Phi(t)$ at time t, instantaneous cost

$$c(y) = \begin{cases} \frac{1}{2\alpha_+} y^2, & y \geq 0 \\ \frac{1}{2\alpha_-} y^2, & y < 0 \end{cases}$$

- Short is more costly than long: $\alpha_- \leq \alpha_+$

Existence

Theorem: • There exists a unique equilibrium price $P(t) = v(t, X(t))$, and v is the solution of

$$v_t(t,x) + \sup_{I \subseteq \{1,\dots,n\}} \left\{ \tfrac{1}{2}\Sigma_I^2(t,x) v_{xx}(t,x) - \kappa_I(t,x) \right\} = 0, \quad v(T,\cdot) = f,$$

where the coefficients are defined as

$$\kappa_I(t,x) = \frac{s(t,x)}{|I|\alpha_- + |I^c|\alpha_+},$$

$$\Sigma_I^2(t,x) = \tfrac{\alpha_-}{|I|\alpha_- + |I^c|\alpha_+} \sum_{i \in I} \sigma_i^2(t,x) + \tfrac{\alpha_+}{|I|\alpha_- + |I^c|\alpha_+} \sum_{i \in I^c} \sigma_i^2(t,x)$$

• The optimal portfolios $\Phi_i(t) = \phi_i(t, X(t))$ are unique and given by

$$\phi_i(t,x) = \alpha_{\mathrm{sign}(\mathcal{L}^i v(t,x))} \mathcal{L}^i v(t,x), \quad \mathcal{L}^i v(t,x) = \partial_t v(t,x) + \tfrac{1}{2}\sigma_i^2 \partial_{xx} v(t,x).$$

Control Representation

- HJB equation of the control problem

$$v(t,x) = \sup_{\mathcal{I}\in\Theta} E\left[f(X_{\mathcal{I}}^{t,x}(T)) - \int_0^T \kappa_{\mathcal{I}(r)}(r, X_{\mathcal{I}}^{t,x}(r))\, dr \right]$$

  - Θ is the set of $2^{\{1,\dots,n\}}$-valued, progressive processes
  - $X_{\mathcal{I}}^{t,x}(r)$, $r \in [t, T]$ is the solution of

$$dX(r) = \Sigma_{\mathcal{I}(r)}(r, X(r))\, dW(t), \quad X(t) = x.$$

- Interpretation?

A Principal Agent Problem

- At each state (t, x), principal assigns a cost coefficient $\alpha_i \in \{\alpha_-, \alpha_+\}$ to every agent $i \in \{1, \dots, n\}$
- This assignment will play the role of a contract (Second Best)
- With these coefficients given, agents maximize

$$E_i \left[\int_0^T \Phi(t)\, dP(t) - \int_0^T c_i(t, X(t), \Phi(t))\, dt \right]$$

where $c_i(t, x, y) = \alpha_i(t, x) y^2$ irrespectively of y being long or short.

- An assignment can be summarized as a set

$$I(t, x) = \{i \in \{1, \dots, n\} : \alpha_i(t, x) = \alpha_-\}.$$

I.e., $I = \{\text{agents with } \alpha_-\}$, $I^c = \{\text{agents with } \alpha_+\}$

Principal Agent Problem: Solution

Theorem:

(i) For any assignment $\mathcal{I}(t) = I(t, X(t))$ of the principal, there exists a unique equilibrium price $P_{\mathcal{I}}(t) = v_{\mathcal{I}}(t, X(t))$, and

$$v_{\mathcal{I}}(t,x) = E\left[f(X_{\mathcal{I}}^{t,x}(T)) - \int_0^T \kappa_{\mathcal{I}(r)}(r, X_{\mathcal{I}}^{t,x}(r))\,dr\right]$$

(ii) If the principal's aim is to maximize the price,

- the optimal value is our previous equilibrium price $v(t,x)$
- the optimal contract assigns, in equilibrium, α_- to short positions and α_+ to long positions

$\rightarrow$ Interpretation for Σ_I, κ_I in our PDE for $v(t,x)$

Comparative Statics and Limiting Cases

- The price is decreasing wrt. supply
- The price is increasing wrt. α_+ (when α_- is fixed)
- The price is decreasing wrt. α_- (when α_+ is fixed)

Infinite Cost for Short: As $\alpha_- \to 0$, the price v^{α_-,α_+} converges to the price from Part II:

$$v_t + \sup_{\emptyset \neq J \subseteq \{1,\dots,n\}} \left\{ \frac{1}{2} \frac{1}{|J|} \sum_{i \in J} \sigma_i^2 v_{xx} - \frac{s}{|J|\alpha_+} \right\} = 0$$

Zero Cost for Long: As $\alpha_+ \to \infty$, the price v^{α_-,α_+} converges to the price from Part I:

$$v_t + \sup_{i \in \{1,\dots,n\}} \frac{1}{2} \sigma_i^2 v_{xx} = 0$$

In particular, the limit is independent of α_- and s

Comparison of Dynamic and Static Models

- Again, we can compare with the static version

- Resale and delay options now apply to long and short positions
- The resale option for short positions depresses the dynamic price
- "Bubble" may have either sign

- In the limits

$$\alpha_+ \to \infty \qquad \text{and/or} \qquad \alpha_- \to 0 \text{ and } s \to 0,$$

  the bubble is always nonnegative, as in Part I
- Main difference to previous models: increasing marginal cost of carry

Conclusion

Part I:

- Resale option leads to UVM price and speculative bubble

Parts II–III: A tractable model where

- Supply affects the price as a running cost
- Delay effect can depress the dynamic equilibrium price
- Short-selling is possible and may further depress the price

Thank you

Conclusion

Part I:

- Resale option leads to UVM price and speculative bubble

Parts II–III: A tractable model where

- Supply affects the price as a running cost
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Thank you