A construction of the affine VW supercategory

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Background: vector superspaces. Work over $\mathbb{C}$.

A $\mathbb{Z}_2$-graded vector space $V = V_{\overline{0}} \oplus V_{\overline{1}}$ is a *vector superspace*.

The *superdimension* of V is

$$\dim(V) := (\dim V_{\overline{0}} | \dim V_{\overline{1}}) = \dim V_{\overline{0}} - \dim V_{\overline{1}}.$$

Given a homogeneous element $v \in V$, the *parity* (or the *degree*) of v is denoted by $\overline{v} \in \{\overline{0}, \overline{1}\}$.

The *parity switching functor* π sends $V_{\overline{0}} \mapsto V_{\overline{1}}$ and $V_{\overline{1}} \mapsto V_{\overline{0}}$.

Let $m = \dim V_{\overline{0}}$ and $n = \dim V_{\overline{1}}$.

A *Lie superalgebra* is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a Lie superbracket (supercommutator) $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies super skew symmetry

$$[x, y] = xy - (-1)^{\bar{x}\bar{y}} yx = -(-1)^{\bar{x}\bar{y}} [y, x]$$

and super Jacobi identity

$$[x, [y, z]] = [[x, y], z] + (-1)^{\bar{x}\bar{y}} [y, [x, z]],$$

for x, y, and z homogeneous.

Now, given a homogeneous ordered basis for

$$V = \underbrace{\mathbb{C}\{v_1, \ldots, v_m\}}_{V_{\bar{0}}} \oplus \underbrace{\mathbb{C}\{v_{1'}, \ldots, v_{n'}\}}_{V_{\bar{1}}},$$

the *Lie superalgebra* is the endomorphism algebra $\mathrm{End}_{\mathbb{C}}(V)$ explicitly given by

Matrix representation for $\mathfrak{gl}(m|n)$.

$$\mathfrak{gl}(m|n) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \in M_{m,m}, \ \ B, C^t \in M_{m,n}, \ \ D \in M_{n,n} \right\},$$

where $M_{i,j} := M_{i,j}(\mathbb{C})$.

Since $\mathfrak{gl}(m|n) = \mathfrak{gl}(m|n)_{\bar{0}} \oplus \mathfrak{gl}(m|n)_{\bar{1}}$,

$$\mathfrak{gl}(m|n)_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{gl}(m|n)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}.$$

We say $\mathfrak{gl}(m|n)$ is the *general linear Lie superalgebra*, and V is the *natural representation* of $\mathfrak{gl}(m|n)$.

The grading on $\mathfrak{gl}(m|n)$ is induced by V.

Periplectic Lie superalgebras $\mathfrak{p}(n)$.

Let $m = n$. Then

$$V = \mathbb{C}^{2n} = \underbrace{\mathbb{C}\{v_1, \ldots, v_n\}}_{V_{\bar{0}}} \oplus \underbrace{\mathbb{C}\{v_{1'}, \ldots, v_{n'}\}}_{V_{\bar{1}}}.$$

Define $\beta : V \otimes V \to \mathbb{C} = \mathbb{C}_{\bar{0}}$ as an odd, symmetric, nondegenerate bilinear form satisfying:

$$\beta(v, w) = \beta(w, v), \qquad \beta(v, w) = 0 \quad \text{if } \overline{v} = \overline{w}.$$

That is, β satisfies

$$\beta(v, w) = (-1)^{\bar{v}\bar{w}} \beta(w, v).$$

We define *periplectic (strange) Lie superalgebras* as:

$$\mathfrak{p}(n) := \{x \in \mathrm{End}_{\mathbb{C}}(V) : \beta(xv, w) + (-1)^{\overline{xv}} \beta(v, xw) = 0\}.$$

In terms of the above basis,

$$\mathfrak{p}(n) = \left\{ \left(\begin{array}{cc} A & B \\ C & -A^t \end{array} \right) \in \mathfrak{gl}(n|n) : B = B^t, C = -C^t \right\},$$

where

$$\mathfrak{p}(n)_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \right\} \cong \mathfrak{gl}_n(\mathbb{C}) \quad \text{and} \quad \mathfrak{p}(n)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}.$$

Symmetric monoidal structure.

Consider the category $\mathcal{C}$ of representations of $\mathfrak{p}(n)$, where

$$\mathrm{Hom}_{\mathfrak{p}(n)}(V, V') := \{f : V \to V' : f \text{ homogeneous}, \mathbb{C}\text{-linear}, \\ f(x.v) = (-1)^{\overline{xf}} x.f(v), v \in V, x \in \mathfrak{p}(n)\}.$$

Then the universal enveloping algebra $U(\mathfrak{p}(n))$ is a Hopf superalgebra:

- (coproduct) $\Delta(x) = x \otimes 1 + 1 \otimes x$,
- (counit) $\epsilon(x) = 0$,
- (antipode) S(x) = -x.

So $\mathcal{C}$ is a monoidal category.
Now for $x \otimes y \in U(\mathfrak{p}(n)) \otimes U(\mathfrak{p}(n))$ on $v \otimes w$,

$$(x \otimes y).(v \otimes w) = (-1)^{\overline{yv}} xv \otimes yw.$$

Symmetric monoidal structure.

For $x, y, a, b \in U(\mathfrak{p}(n))$, multiplication is defined as

$$(x \otimes y) \circ (a \otimes b) := (-1)^{\overline{ya}}(x \circ a) \otimes (y \circ b),$$

and for two representations V and V', the *super swap*

$$\sigma : V \otimes V' \longrightarrow V' \otimes V, \quad \sigma(v \otimes w) = (-1)^{\overline{vw}} w \otimes v$$

is a map of $\mathfrak{p}(n)$-representations whose dual satisfies $\sigma^* = -\sigma$. Thus $\mathcal{C}$ is a symmetric monoidal category.

Furthermore, β induces an identification between V and its dual V^* via $V \to V^*, \quad v \mapsto \beta(v, -),$ identifying $V_{\overline{1}}$ with $V_{\overline{0}}^*$ and $V_{\overline{0}}$ with $V_{\overline{1}}^*$.

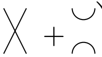
This induces the dual map (where $\overline{\beta} = \overline{\beta^*} = 1$)

$$\beta^* : \mathbb{C} \cong \mathbb{C}^* \longrightarrow (V \otimes V)^* \cong V \otimes V, \quad \beta^*(1) = \sum_i v_{i'} \otimes v_i - v_i \otimes v_{i'}.$$

Quadratic (fake) Casimir Ω & Jucys-Murphy elements y_ℓ's.

Now, define

$$\Omega := 2 \sum_{x \in \mathcal{X}} x \otimes x^* \in \mathfrak{p}(n) \otimes \mathfrak{gl}(n|n) \qquad \Big(2\Omega = \Big),$$



where $\mathcal{X}$ is a basis of $\mathfrak{p}(n)$ and $x^* \in \mathfrak{p}(n)^*$ is a dual basis element of $\mathfrak{p}(n)$, with $\mathfrak{p}(n)^* = \mathfrak{p}(n)^\perp$, taken with respect to the supertrace:

$$\operatorname{str}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \operatorname{tr}(A) - \operatorname{tr}(D).$$

The actions of Ω and $\mathfrak{p}(n)$ commute on $M \otimes V$, so Ω is in the centralizer $\operatorname{End}_{\mathfrak{p}(n)}(M \otimes V)$.

We define

$$Y_\ell : M \otimes V^{\otimes a} \longrightarrow M \otimes V^{\otimes a} \quad \text{as} \quad Y_\ell := \sum_{i=0}^{\ell-1} \Omega_{i,\ell} =$$

,

where $\Omega_{i,\ell}$ acts on the i-th and ℓ-th factor, and identity otherwise, where the 0-th factor is the module M.

Review: classical Schur-Weyl duality.

Let W be an n-dimensional complex vector space. Consider $W^{\otimes a}$. Then the symmetric group S_a acts on $W^{\otimes a}$ by permuting the factors: for $s_i = (i \;\; i+1) \in S_a$,

$$s_i.(w_1 \otimes \cdots \otimes w_a) = w_1 \otimes \cdots \otimes w_{i+1} \otimes w_i \otimes \cdots \otimes w_a.$$

We also have the full linear group $GL(W)$ acting on $W^{\otimes a}$ via the diagonal action: for $g \in GL(W)$,

$$g.(w_1 \otimes \cdots \otimes w_a) = gw_1 \otimes \cdots \otimes gw_a.$$

Then actions of $GL(W)$ (left natural action) and S_a (right permutation action) commute giving us the following:

Classical Schur-Weyl duality.

Consider the natural representations

$$(\mathbb{C}S_a)^{op} \xrightarrow{\phi} \mathrm{End}_{\mathbb{C}}(W^{\otimes a}) \quad \text{and} \quad GL(W) \xrightarrow{\psi} \mathrm{End}_{\mathbb{C}}(W^{\otimes a}).$$

Then Schur-Weyl duality gives us

1. $\phi(\mathbb{C}S_a) = \mathrm{End}_{GL(W)}(W^{\otimes a})$,
2. if $n \geq a$, then ϕ is injective. So $\mathrm{im}\, \phi \cong \mathrm{End}_{GL(W)}(W^{\otimes a})$,
3. $\psi(GL(W)) = \mathrm{End}_{\mathbb{C}S_a}(W^{\otimes a})$,
4. there is an irreducible $(GL(W), (\mathbb{C}S_a)^{op})$-bimodule decomposition (see next slide):

Classical Schur-Weyl duality (continued).

$$W^{\otimes a} = \bigoplus_{\substack{\lambda=(\lambda_1,\lambda_2,\ldots)\vdash a \\ \ell(\lambda)\leq n}} \Delta_\lambda \otimes S^\lambda,$$

where

- Δ_λ is an irreducible $GL(W)$-module associated to the partition λ,
- S^λ is an irreducible $\mathbb{C}S_a$ (Specht) module associated to λ, and
- $\ell(\lambda) = \max\{i \in \mathbb{Z} : \lambda_i \neq 0, \lambda = (\lambda_1, \lambda_2, \ldots)\}$.

In the above setting, we say $\mathbb{C}S_a$ and $GL(W)$ in $\mathrm{End}_{\mathbb{C}}(W^{\otimes a})$ are centralizers of one another.

Other cases of Schur-Weyl duality.

For the orthogonal group $O(n)$ and symplectic group Sp_{2n}, the symmetric group S_n should be replaced by a Brauer algebra.

A *Brauer algebra* $\mathrm{Br}_a^{(x)}$ with a parameter $x \in \mathbb{C}$ is a unital $\mathbb{C}$-algebra with generators $s_1, \ldots, s_{a-1}, e_1, \ldots, e_{a-1}$ and relations:

$$
\begin{aligned}
s_i^2 = 1, \quad e_i^2 = xe_i, \quad e_i s_i = e_i = s_i e_i \quad & \text{for all } 1 \leq i \leq a-1, \\
s_i s_j = s_j s_i, \quad s_i e_j = e_j s_i, \quad e_i e_j = e_j e_i \quad & \text{for all } 1 \leq i < j-1 \leq a-2, \\
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad & \text{for all } 1 \leq i \leq a-2, \\
e_i e_{i+1} e_i = e_i, e_{i+1} e_i e_{i+1} = e_{i+1} \quad & \text{for all } 1 \leq i \leq a-2, \\
s_i e_{i+1} e_i = s_{i+1} e_i, \quad e_{i+1} e_i s_{i+1} = e_{i+1} s_i \quad & \text{for all } 1 \leq i \leq a-2.
\end{aligned}
$$

The group ring of the Brauer algebra $\mathrm{Br}_a^{(n)}$ and $O(n)$ in $\mathrm{End}(W^{\otimes a})$ centralize one another, where $\dim W = n$,

and

the group ring of the Brauer algebra $\mathrm{Br}_a^{(-2n)}$ and Sp_{2n} in $\mathrm{End}(V^{\otimes a})$ centralize one another, where $\dim V = 2n$.

Now, in **higher** Schur-Weyl duality, we construct a result analogous to

$$\mathbb{C}S_a \cong \mathrm{End}_{GL(W)}(W^{\otimes a}),$$

but we use the existence of *commuting actions* on the tensor product of arbitrary $\mathfrak{gl}_n$-representation M with $W^{\otimes a}$:

$$\mathfrak{gl}_n \circlearrowright M \otimes W^{\otimes a} \circlearrowleft H_a,$$

where H_a is the *degenerate affine Hecke algebra*, i.e., it is a deformation of the symmetric group S_a.

The algebra H_a has generators $s_1, \dots, s_{a-1}, y_1, \dots, y_a$ and relations

$$\begin{aligned} s_i^2 &= 1, \\ s_i s_j &= s_j s_i \quad \text{whenever } |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ y_i y_j &= y_j y_i, \\ y_i s_j &= s_j y_i \quad \text{whenever } i-j \neq 0, 1, \\ y_{i+1} s_i &= s_i y_i + 1. \end{aligned}$$

The Hecke algebra H_a contains the symmetric algebra $\mathbb{C}S_a$ and the polynomial algebra $\mathbb{C}[y_1, \ldots, y_a]$ as subalgebras.

So as a vector space, $H_a \cong \mathbb{C}S_a \otimes \mathbb{C}[y_1, \ldots, y_a]$, and has a basis

$$\mathcal{B} = \{wy_1^{k_1} \cdots y_a^{k_a} : w \in S_a, k_i \in \mathbb{N}_0\}.$$

Our goal: construct higher Schur-Weyl duality for $\mathfrak{p}(n)$.

That is, *construct* another algebra whose action on $M \otimes V^{\otimes a}$ commutes with the action of $\mathfrak{p}(n)$.

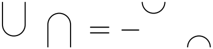
This algebra is precisely the degenerate affine Brauer superalgebra $s\mathbb{W}_a$.

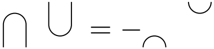
Degenerate affine Brauer superalgebras (generators and local moves).

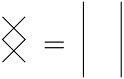
$s\mathbb{W}_a$ has generators s_i, b_i, b_i^*, y_j, where $i = 1, \ldots, a-1$, $j = 1, \ldots, a$ and relations

Continued in the next slide.

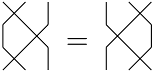
Degenerate affine Brauer superalgs (local moves).



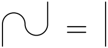




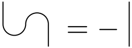
(braid reln)



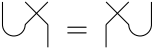
(braid reln)



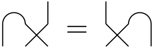
(adjunction)

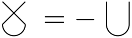


(adjunction)

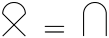


(untwisting reln)

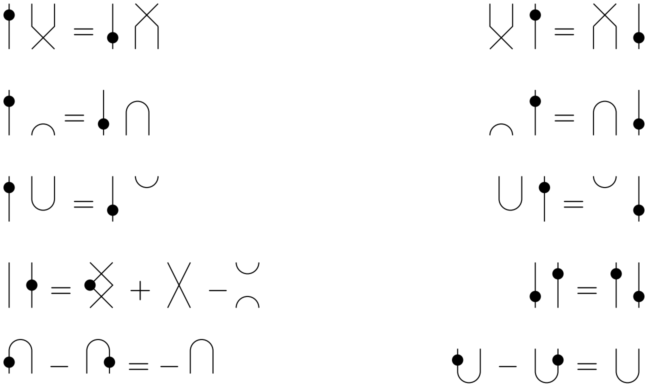




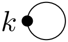
(untwisting reln)



Degenerate affine Brauer superalgs (local moves).



Lemma. For any $k \geq 0$,



$= 0.$

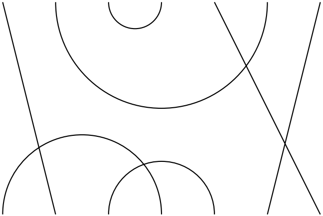
Normal diagrams.

Call a diagram $d \in \mathrm{Hom}_{s\mathcal{B}r}(a, b)$ *normal* if all of the following hold:

- any two strings intersect at most once;
- no string intersects itself;
- no two cups or caps are at the same height;
- all cups are above all caps;
- the height of caps decreases when the caps are ordered from left to right with respect to their left ends;
- the height of cups increases when the cups are ordered from left to right with respect to their left ends.

Every string in a normal diagram has either one cup, or one cap, or no cups and caps, and there are no closed loops. A diagram with no loops in $\mathrm{Hom}_{s\mathcal{B}r}(a, b)$ has $\frac{a+b}{2}$ strings. In particular, if $a + b$ is odd then the Hom-space is zero.

Example: normal diagram in the signed Brauer algebra $s\mathcal{B}r_a$.



Algebraically, it is written as $s_2 s_3 s_5 b_2^* b_2 b_4^* b_4 s_1 s_3 s_6$.

The monomial corresponding to a normal diagram is called a *regular monomial*.

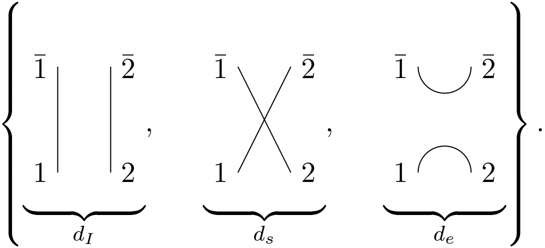
Connectors.

Each normal diagram $d \in \mathrm{Hom}_{s\mathcal{B}r}(a,b)$, where $a, b \in \mathbb{N}_0$, gives rise to a partition $P(d)$ of the set of $a+b$ points into 2-element subsets given by the endpoints of the strings in d.

We call such a partition a *connector*, and write $\mathrm{Conn}(a,b)$ as the set of all such connectors. Its size is $(a+b-1)!!$.

Example. Let $a = b = 2$. Label the endpoints along the bottom row of d as 1 and 2 (reading from left to right), and label the endpoints along the top row of d as $\bar{1}$ and $\bar{2}$ (reading from left to right). Then

$$\mathrm{Hom}_{s\mathcal{B}r}(2,2) =$$



Three possible connectors for a diagram in $\mathrm{Hom}_{s\mathcal{B}r}(2,2)$:

$$P(d_I) = \{\{1, \bar{1}\}, \{2, \bar{2}\}\},$$
$$P(d_s) = \{\{1, \bar{2}\}, \{2, \bar{1}\}\},$$
$$P(d_e) = \{\{1, 2\}, \{\bar{1}, \bar{2}\}\},$$

and $\mathrm{Conn}(2,2) = \{P(d_I), P(d_s), P(d_e)\}$.

For each connector $c \in \mathrm{Conn}(a, b)$, we pick a normal diagram $d_c \in P^{-1}(c) \subset \mathrm{Hom}_{s\mathcal{B}r}(a, b)$.

Remark. Different normal diagrams in a single fibre $P^{-1}(c)$ differ only by braid relations, and thus represent the same morphism.

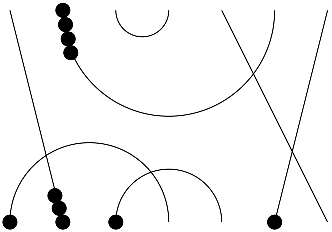
Theorem (BDEHHILNSS)

The set $S_{a,b} = \{d_c : c \in \mathrm{Conn}(a,b)\}$ *is a basis of* $\mathrm{Hom}_{s\mathcal{B}r}(a,b)$.

A dotted diagram $d \in \mathrm{Hom}_{s\mathbb{W}}(a,b)$ is *normal* if:

- the underlying diagram obtained by erasing the dots is normal;
- all dots on cups and caps are on the leftmost end, and all dots on the through strings are at the bottom.

Example. A normal diagram in $\mathrm{Hom}_{s\mathbb{W}}(7,7)$:



Algebraically, it is written as $y_2^4 s_2 s_3 s_5 b_2^* b_2 b_4^* b_4 s_1 s_3 s_6 y_1 y_2^3 y_3 y_6$.

Normal dotted diagrams.

Let $S^{\bullet}_{a,b}$ be the normal dotted diagrams obtained by taking all diagrams in $S_{a,b}$ and adding dots to them in all possible ways.

Let $S^{\leq k}_{a,b} \subseteq S^{\bullet}_{a,b}$ be the diagrams with at most k dots.

Theorem (Basis theorem, BDEHHILNSS)

The set $S^{\leq k}_{a,b}$ is a basis of $\mathrm{Hom}_{s\mathbb{W}}(a,b)^{\leq k}$, and the set $S^{\bullet}_{a,b}$ is a basis of $\mathrm{Hom}_{s\mathbb{W}}(a,b)$.

Our affine VW superalgebra $s\mathbb{W}_a$ is:

- super (signed) version of the degenerate BMW algebra,
- the signed version of the affine VW algebra, and
- an affine version of the Brauer superalgebra.

The center of $s\mathbb{W}_a = \mathrm{End}_{s\mathbb{W}}(a)$, $a \geq 2 \in \mathbb{N}$.

Theorem (BDEHHILNSS)

The center $Z(s\mathbb{W}_a)$ consists of all polynomials of the form

$$\prod_{1 \leq i < j \leq a} ((y_i - y_j)^2 - 1)\widetilde{f} + c,$$

where $\widetilde{f} \in \mathbb{C}[y_1, \ldots, y_a]^{S_a}$ and $c \in \mathbb{C}$.

The deformed squared Vandermonde determinant $\prod_{1 \leq i < j \leq a}((y_i - y_j)^2 - 1)$ is symmetric, so

$$\prod_{1 \leq i < j \leq a} ((y_i - y_j)^2 - 1) \in \mathbb{C}[y_1, \ldots, y_a]^{S_a}.$$

Affine VW supercategory sW and connections to Brauer supercategory $s\mathcal{B}r$.

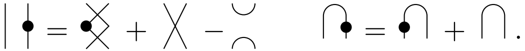
The affine VW supercategory (or the affine Nazarov-Wenzl supercategory) is the $\mathbb{C}$-linear strict monoidal supercategory generated as a monoidal supercategory by a single object $\bigstar$, morphisms

$s =$  $: \bigstar \otimes \bigstar \longrightarrow \bigstar \otimes \bigstar$, $\flat =$  $: \bigstar \otimes \bigstar \to \mathbf{1}$,

$\flat^* =$  $: \mathbf{1} \to \bigstar \otimes \bigstar$, and an additional morphism

$y =$  $: \bigstar \otimes \bigstar \longrightarrow \bigstar \otimes \bigstar$, subject to the braid, snake (adjunction),

and untwisting relations, and the dot relations:



Objects in sW can be identified with natural numbers, identifying $a \in \mathbb{N}_0$ with $\bigstar^{\otimes a}$, $\bigstar^{\otimes 0} = \mathbf{1}$, and the morphisms are linear combinations of dotted diagrams.

$s\mathbb{W}$ and $s\mathcal{B}r$.

The category $s\mathbb{W}$ can alternatively be generated by vertically stacking $\flat_i$, $\flat_i^*$, s_i, and $y_i = 1_{i-1} \otimes y \otimes 1_{a-i} \in \mathrm{Hom}_{s\mathbb{W}}(a, a)$.

It is a filtered category, i.e., the hom spaces $\mathrm{Hom}_{s\mathbb{W}}(a, b)$ have a filtration by the span $\mathrm{Hom}_{s\mathbb{W}}(a, b)^{\leq k}$ of all dotted diagrams with at most k dots.

The Brauer supercategory $s\mathcal{B}r$ is the $\mathbb{C}$-linear strict monoidal supercategory generated as a monoidal supercategory by a single object $\bigstar$, and morphisms $s = \times : \bigstar \otimes \bigstar \longrightarrow \bigstar \otimes \bigstar$, $\flat = \frown : \bigstar \otimes \bigstar \to \mathbf{1}$, and $\flat^* = \smile : \mathbf{1} \to \bigstar \otimes \bigstar$, subject to the relations above.

If M is the trivial representation, then actions on $s\mathbb{W}$ factor through $s\mathcal{B}r$.

Thank you. Questions?

The algebra $A_\hbar$ and its specializations A_t, where $t \in \mathbb{C}$.

Definition

Let $A_\hbar$ be the superalgebra over $\mathbb{C}[\hbar]$ with generators $s_i,\ e_i,\ y_j$ for $1 \leq i \leq a-1,\ 1 \leq j \leq a$, where $\overline{s_i} = \overline{e_i} = \overline{y_j} = 0$, subject to the relations:

1. Involutions: $s_i^2 = 1$ for $1 \leq i < a$.
2. Commutation relations:
   1. $s_i e_j = e_j s_i$ if $|i-j| > 1$,
   2. $e_i e_j = e_j e_i$ if $|i-j| > 1$,
   3. $e_i y_j = y_j e_i$ if $j \neq i, i+1$,
   4. $y_i y_j = y_j y_i$ for $1 \leq i, j \leq a$.
3. Affine braid relations:
   1. $s_i s_j = s_j s_i$ if $|i-j| > 1$,
   2. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \leq i \leq a-1$,
   3. $s_i y_j = y_j s_i$ if $j \neq i, i+1$.
4. Snake relations:
   1. $e_{i+1} e_i e_{i+1} = -e_{i+1}$,
   2. $e_i e_{i+1} e_i = -e_i$ for $1 \leq i \leq a-2$.
5. Tangle and untwisting relations:
   1. $e_i s_i = e_i$ and $s_i e_i = -e_i$ for $1 \leq i \leq a-1$,
   2. $s_i e_{i+1} e_i = s_{i+1} e_i$,
   3. $s_{i+1} e_i e_{i+1} = -s_i e_{i+1}$,
   4. $e_{i+1} e_i s_{i+1} = e_{i+1} s_i$,
   5. $e_i e_{i+1} s_i = -e_i s_{i+1}$ for $1 \leq i \leq a-2$.
6. Idempotent relations: $e_i^2 = 0$ for $1 \leq i \leq a-1$.
7. Skein relations:
   1. $s_i y_i - y_{i+1} s_i = -\hbar e_i - \hbar$,
   2. $y_i s_i - s_i y_{i+1} = \hbar e_i - \hbar$ for $1 \leq i \leq a-1$.
8. Unwrapping relations: $e_1 y_1^k e_1 = 0$ for $k \in \mathbb{N}$.
9. (Anti)-symmetry relations:
   1. $e_i (y_{i+1} - y_i) = \hbar e_i$,
   2. $(y_{i+1} - y_i) e_i = -\hbar e_i$ for $1 \leq i \leq a-1$.

For $t \in \mathbb{C}$, let A_t be the quotient of $A_\hbar$ by the ideal generated by $\hbar - t$.

A sketch of proof of the Theorem on slide 27.

1. The filtered algebra $s\mathbb{W}_a$ (via the filtration by the degree of the polynomials in $\mathbb{C}[y_1, \ldots, y_a]$) is a Poincaré-Birkhoff-Witt (PBW) deformation of the associated graded superalgebra $gs\mathbb{W}_a = \mathrm{gr}(s\mathbb{W}_a)$,
2. For $\hbar$ a parameter, the Rees construction gives the algebra $A_\hbar$ over $\mathbb{C}[\hbar]$ such that the specializations $\hbar = 1$ and $\hbar = 0$ are precisely $A_1 = s\mathbb{W}_a$ and $A_0 = gs\mathbb{W}_a$,
3. Describe the center of the $\mathbb{C}[\hbar]$-algebra $A_\hbar$, and all its specializations A_t for any $t \in \mathbb{C}$ using the Basis Theorem,
4. Determine the center of $gs\mathbb{W}_a$ using the isomorphism $\mathrm{Rees}(Z(A_1)) \cong Z(\mathrm{Rees}(A_1)) \cong Z(A_\hbar)$, and
5. Find a lift of the appropriate basis elements to $s\mathbb{W}_a$ to obtain the center of $s\mathbb{W}_a$.

Expanding on 2.

Let $B = \bigcup_{k\geq 0} B^{\leq k}$ be a filtered $\mathbb{C}$-algebra. The *Rees algebra* of B is the $\mathbb{C}[\hbar]$-algebra $\mathrm{Rees}(B)$, given as a $\mathbb{C}$-vector space by $\mathrm{Rees}(B) = \bigoplus_{k\geq 0} B^{\leq k}\hbar^k$, with multiplication and the $\hbar$-action given by

$$(a\hbar^i)(b\hbar^j) = (ab)\hbar^{i+j} \text{ for } a \in B^{\leq i}, b \in B^{\leq j}, \text{ and } ab \in B^{\leq i+j},$$

the product in B. It is graded as a $\mathbb{C}$-algebra by the powers of $\hbar$.

Lemma

1. *Let $\bigcup_{i\geq 0} S_i$ be a basis of B compatible with the filtration, where S_i's are pairwise disjoint, and $\bigcup_{i=0}^{k} S_i$ is a basis of $B^{\leq k}$. Then $\bigcup_{i\geq 0} S_i\hbar^i$ is a $\mathbb{C}[\hbar]$-basis of $\mathrm{Rees}(B)$.*
2. $Z(\mathrm{Rees}(B)) = \mathrm{Rees}(Z(B))$.
3. $\mathrm{Rees}(A_1) \cong A_\hbar$, *an isomorphism of $\mathbb{C}[\hbar]$-algebras.*

Expanding on 3.

Show that $Z(A_\hbar) \subseteq \mathbb{C}[\hbar][y_1, \ldots, y_a]^{S_a}$.

Lemma

For $f \in A_\hbar$, the following are equivalent:

- (a) $fy_i = y_i f$ *for all* $i \in [a] = \{1, 2, \ldots, a\}$;
- (b) $f \in \mathbb{C}[\hbar][y_1, \ldots, y_a]$.

So $Z(A_\hbar) \subseteq \mathbb{C}[\hbar][y_1, \ldots, y_a]$.

Lemma. Let $f \in \mathbb{C}[\hbar][y_1, \ldots, y_a] \subseteq A_\hbar$ and $1 \leq i \leq a-1$.

- (a) If $fs_i = s_i f$, then $f(y_1, \ldots, y_i, y_{i+1}, \ldots, y_a) = f(y_1, \ldots, y_{i+1}, y_i, \ldots, y_a)$.
- (b) For the special value $\hbar = 0$, the converse also holds: if $f(y_1, \ldots, y_i, y_{i+1}, \ldots, y_a) = f(y_1, \ldots, y_{i+1}, y_i, \ldots, y_a)$, then $fs_i = s_i f$ in A_0.

So $Z(A_\hbar)$ is a subalgebra of $\mathbb{C}[\hbar][y_1, \ldots, y_a]^{S_a}$.

Expanding on 3 (continued).

Consider the following elements in $\mathbb{C}[\hbar][y_1, \ldots, y_a]$:

$$z_{ij} = (y_i - y_j)^2, \text{ for } 1 \leq i \neq j \leq a \quad \text{and} \quad D_\hbar = \prod_{1 \leq i < j \leq a} (z_{ij} - \hbar^2),$$

where $D_\hbar$ is symmetric. So $D_\hbar \in \mathbb{C}[\hbar][y_1, \ldots, y_a]^{S_a}$.
Use $D_\hbar$ to produce central elements in $A_\hbar$.

Lemma

1. *For any* $1 \leq i \leq a-1$, $e_i \cdot (z_{i,i+1} - \hbar^2) = (z_{i,i+1} - \hbar^2) \cdot e_i = 0$ *in* $A_\hbar$*, and consequently* $e_i D_\hbar = D_\hbar e_i = 0$*.*
2. *For any* $1 \leq k \leq a-1$*, we have* $D_\hbar s_k = s_k D_\hbar$*.*
3. *Let* $1 \leq i \leq a-1$*, and let* $\tilde{f} \in \mathbb{C}[\hbar][y_1, \ldots, y_a]$ *be symmetric in* y_i, y_{i+1}*. Then there exist polynomials* $p_j = p_j(y_1, \ldots, y_a) \in \mathbb{C}[\hbar][y_1, \ldots, y_a]$ *such that* $\tilde{f} s_i = s_i \tilde{f} + \sum_{j=0}^{\deg \tilde{f} - 1} y_i^j \cdot e_i \cdot p_j$*.*

Expanding on 3 (continued).

Lemma

Let $\tilde{f} \in \mathbb{C}[\hbar][y_1, \ldots, y_a]^{S_a}$ be an arbitrary symmetric polynomial, and $c \in \mathbb{C}$. Then $f = D_\hbar \tilde{f} + c \in Z(A_\hbar)$.

Expanding on 4.

Proposition. The center $Z(A_0)$ of the graded VW superalgebra gsW_a consists of all $f \in \mathbb{C}[y_1, \ldots, y_a]$ of the form $f = D_0\tilde{f} + c$, for $\tilde{f} \in \mathbb{C}[y_1, \ldots, y_a]^{S_a}$ and $c \in \mathbb{C}$.

Expanding on 5.

Theorem (BDEHHILNSS)

The center $Z(s\mathbb{W}_a)$ of the VW superalgebra $s\mathbb{W}_a = A_1$ consists of all $f \in \mathbb{C}[y_1, \ldots, y_n]$ of the form $f = D_1\tilde{f} + c$, for an arbitrary symmetric polynomial $\tilde{f} \in \mathbb{C}[y_1, \ldots, y_a]^{S_a}$ and $c \in \mathbb{C}$.

Proof.

For any filtered algebra B there exists a canonical injective algebra homomorphism $\varphi : \operatorname{gr} Z(B) \hookrightarrow Z(\operatorname{gr}(B))$, given by $\varphi(f + Z(B)^{\leq(k-1)}) = f + B^{\leq(k-1)}$ for $f \in Z(B)^{\leq k}$. For $B = s\mathbb{W}_a$ and $\operatorname{gr}(B) = gs\mathbb{W}_a$, $Z(A_0)$ consists of elements of the form $f = D_0\tilde{f} + c$ for $\tilde{f}$ a symmetric polynomial and c a constant. Since $D_1\tilde{f} + c \in Z(s\mathbb{W}_a)$, we have $\varphi(c) = c$, and for $\tilde{f}$ symmetric and homogeneous of degree k, $\varphi(D_1\tilde{f} + s\mathbb{W}_a^{\leq a(a-1)+k-1}) = D_0\tilde{f}$. Using the above Proposition, we see that every $f \in Z(gs\mathbb{W}_a)$ is in the image of φ, so φ is an isomorphism. □

Expanding on 5 (continued).

Theorem (BDEHHILNSS)

The center $Z(A_\hbar)$ of the superalgebra $A_\hbar$ consists of polynomials $f \in \mathbb{C}[\hbar][y_1, \dots, y_n]$ of the form $f = D_\hbar \tilde{f} + c$, for an arbitrary symmetric polynomial $\tilde{f} \in \mathbb{C}[\hbar][y_1, \dots, y_a]^{S_a}$ and $c \in \mathbb{C}[\hbar]$.

Proof.

The center $Z(A_\hbar)$ is isomorphic to $Z(\mathrm{Rees}(A_1))$, which is also isomorphic to $\mathrm{Rees}(Z(A_1))$. The center $Z(A_1)$ consists of elements of the form $f = D_1 \tilde{f} + c$, with $\tilde{f} \in \mathbb{C}[y_1, \dots, y_a]^{S_a}$ and $c \in \mathbb{C}$. Assume $\tilde{f}$ is homogeneous of degree k. Then $D_1 \tilde{f} \in A_1^{\leq k+a(a-1)}$, which gives an element $D_1 \tilde{f} \hbar^{k+a(a-1)}$ of $\mathrm{Rees}(Z(A_1)) \cong Z(\mathrm{Rees}(A_1))$. We see that $Z(A_\hbar)$ is spanned by constants and the preimages under the isomorphism $A_\hbar \cong \mathrm{Rees}(A_1)$ of elements $D_1 \tilde{f} \hbar^{k+a(a-1)}$, which are equal to $D_\hbar \tilde{f}$. □