Characterization of queer super crystals

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based on joint work with Maria Gillespie, Graham Hawkes, Wencin Poh

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Goal

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- In mathematics: projective representations of the symmetric group
- Queer super Lie algebra
- Highest weight crystals for queer super Lie algebras (Grantcharov et al.)
- Characterization of these crystals and how these discoveries were guided by experimentation with SAGEMATH

Outline

Crystals of type A_n

Abstract crystal of type A_n: nonempty set B together with the maps

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\begin{aligned}
e_i, f_i \colon B &\to B \sqcup \{0\} \qquad (i \in I) \\
\mathrm{wt} \colon B &\to \Lambda
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weight lattice $\Lambda = \mathbb{Z}_{\geqslant 0}^{n+1}$
index set $I = \{1, 2, \ldots, n\}$
simple root $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $\quad \epsilon_i$ i-th standard basis vector of $\mathbb{Z}^{n+1}$

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string lengths for $b \in B$

$$\varphi_i(b) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid f_i^k(b) \neq 0\}$$
$$\varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid e_i^k(b) \neq 0\}$$

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We require:

A1. $f_i b = b'$ if and only if $b = e_i b'$
$\mathrm{wt}(b') = \mathrm{wt}(b) + \alpha_i$

Crystal: A_n example

Example

Standard crystal $\mathcal{B}$ for type A_n:

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n} \boxed{n+1}$$

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- $\mathrm{wt}\left(\boxed{i}\right) = \epsilon_i$
- Highest weight element: $\boxed{1}$

Tensor products

B and C crystals of type A_n

Definition

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- Elements: $b \otimes c := (b, c) \in B \times C$
- Weight map: $\mathrm{wt}(b \otimes c) = \mathrm{wt}(b) + \mathrm{wt}(c)$
- Crystal operators:

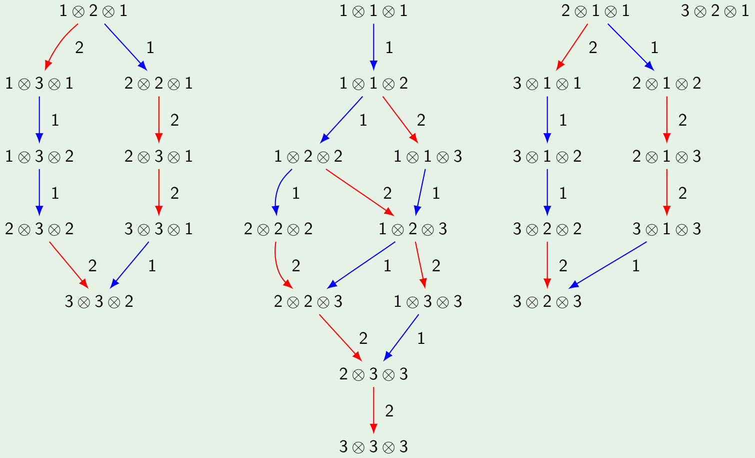
$$f_i(b \otimes c) = \begin{cases} f_i(b) \otimes c & \text{if } \varepsilon_i(b) \geqslant \varphi_i(c) \\ b \otimes f_i(c) & \text{if } \varepsilon_i(b) < \varphi_i(c) \end{cases}$$

$$e_i(b \otimes c) = \begin{cases} e_i(b) \otimes c & \text{if } \varepsilon_i(b) > \varphi_i(c) \\ b \otimes e_i(c) & \text{if } \varepsilon_i(b) \leqslant \varphi_i(c) \end{cases}$$

Example: Tensor product

Example

Components of crystal of words $\mathcal{B}^{\otimes 3} = \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}$ of type A_2:



Motivation

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$$s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$$

$c_{\lambda\mu}^\nu$ = number of highest weights of weight ν in $B(\lambda) \otimes B(\mu)$

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Queer crystal: Developments

- Queer Lie superalgebra $\mathfrak{q}(n)$: a super analogue of $\mathfrak{gl}(n)$

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- [Grantcharov, Jung, Kang, Kashiwara, Kim, '10]:
  Crystal basis theory for queer Lie superalgebras using $U_q(\mathfrak{q}(n))$
  - Introduced queer crystals on words with tensor product rule.
  - Explicit combinatorial realization of queer crystals using semistandard decomposition tableaux.
  - Existence of fake highest (and lowest) weights on queer crystals.

Standard crystal and tensor product

Example

Standard queer crystal $\mathcal{B}$ for $\mathfrak{q}(n+1)$

$$\begin{tikzcd}
\boxed{1} \arrow[r, shift left, "1"] \arrow[r, shift right, dashed, "-1"'] & \boxed{2} \arrow[r, "2"] & \boxed{3} \arrow[r, "3"] & \cdots \arrow[r, "n"] & \boxed{n+1}
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Tensor product: $b \otimes c \in B \otimes C$

$$f_{-1}(b \otimes c) = \begin{cases} b \otimes f_{-1}(c) & \text{if } \operatorname{wt}(b)_1 = \operatorname{wt}(b)_2 = 0 \\ f_{-1}(b) \otimes c & \text{otherwise} \end{cases}$$

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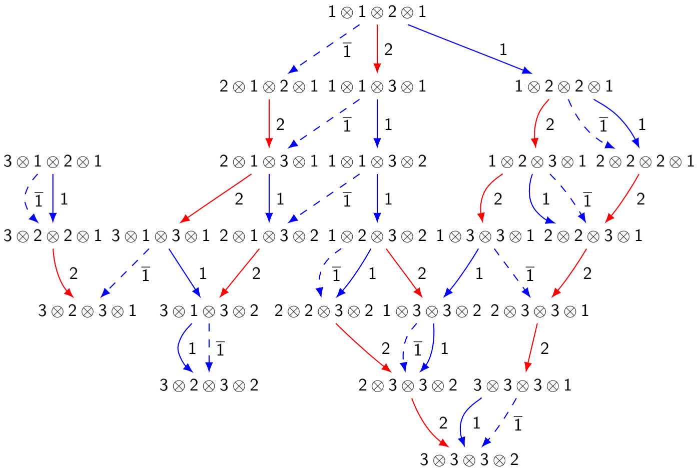
Queer crystal: Example

One connected component of $\mathcal{B}^{\otimes 4}$ for $\mathfrak{q}(3)$:



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In the queer crystal there exist fake highest weight elements.
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where $w_i = s_2 \cdots s_i s_1 \cdots s_{i-1}$ and s_i is the reflection along the i-string

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Theorem (Grantcharov et al. 2014)

Each connected component in $\mathcal{B}^{\otimes \ell}$ has a unique highest weight element with

$$e_i u = 0 \quad \textit{and} \quad e_{-i} u = 0 \quad \textit{for all } i \in I_0 = \{1, 2, \ldots, n\}$$

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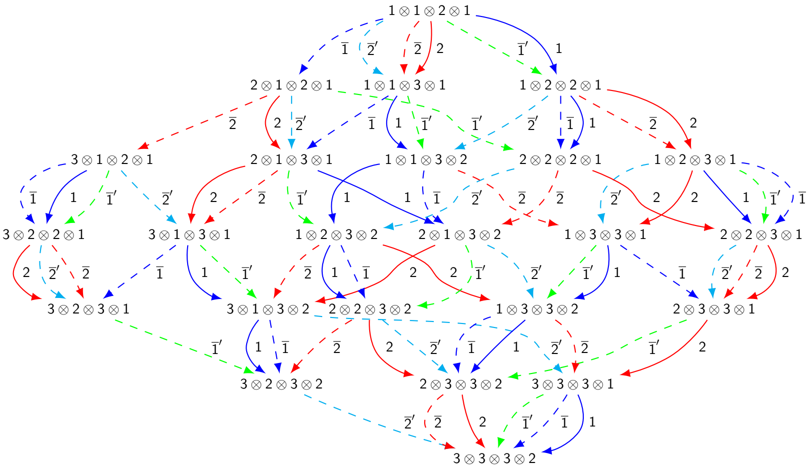
Similarly

$$f_{-i'} := s_{w_0} e_{-(n+1-i)} s_{w_0} \qquad \text{and} \qquad e_{-i'} := s_{w_0} f_{-(n+1-i)} s_{w_0}$$

where w_0 is long word in S_{n+1}, give lowest weight elements.

Queer crystal: Example revisited

Same connected component of $\mathcal{B}^{\otimes 4}$:



Outline

Stembridge axioms

Question

Is there a local characterization of a crystal graph?

Stembridge axioms

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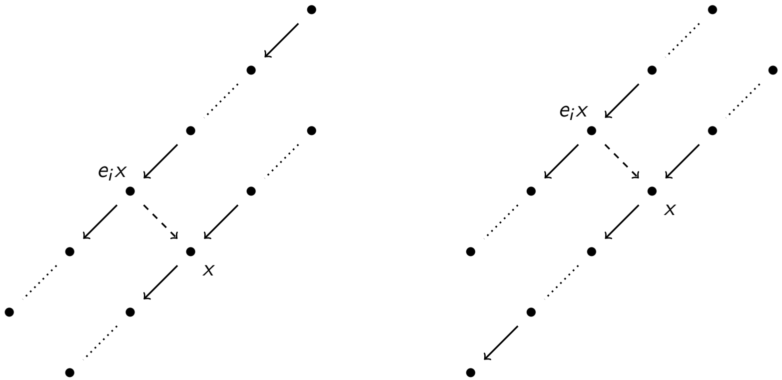
Is there a local characterization of a crystal graph?

- [Stembridge '03] Yes, for crystals of simply-laced root systems (in particular type A_n)
- Local rules characterize Stembridge crystals: allows pure combinatorial analysis of these crystals

Stembridge axioms

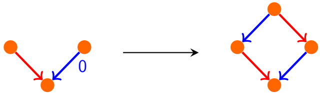
B crystal for a simply-laced root system with index set $I = \{1, 2 \ldots, n\}$.

Axiom S1. For distinct $i, j \in I$ and $x, y \in B$ with $y = e_i x$, then either $\varepsilon_j(y) = \varepsilon_j(x) + 1$ or $\varepsilon_j(y) = \varepsilon_j(x)$.



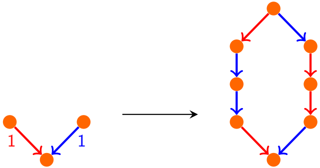
Stembridge axioms

Axiom S2. For distinct $i, j \in I$, if $x \in B$ with both $\varepsilon_i(x) > 0$ and $\varepsilon_j(x) = \varepsilon_j(e_i x) > 0$, then $e_i e_j x = e_j e_i x$ and $\varphi_i(e_j x) = \varphi_i(x)$.



Stembridge axioms

Axiom S3. For distinct $i, j \in I$, if $x \in B$ with both $\varepsilon_j(e_i x) = \varepsilon_j(x) + 1 > 1$ and $\varepsilon_i(e_j x) = \varepsilon_i(x) + 1 > 1$, then $e_i e_j^2 e_i x = e_j e_i^2 e_j x \neq 0$, $\varphi_i(e_j x) = \varphi_i(e_j^2 e_i x)$ and $\varphi_j(e_i x) = \varphi_j(e_i^2 e_j x)$.



Why Stembridge crystals?

Theorem (Stembridge 2003)

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Every connected component of a Stembridge crystal has a unique highest weight element.

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B, B' Stembridge crystals, $u \in B$, $u' \in B'$ highest weight elements
If $\mathrm{wt}(u) = \mathrm{wt}(u')$, then B and B' are isomorphic.

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Stembridge crystals describe the representation theory of the corresponding Lie algebra.

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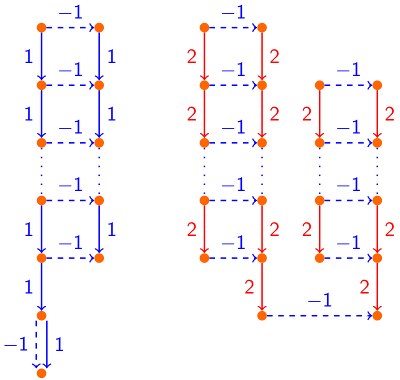
Stembridge type axioms

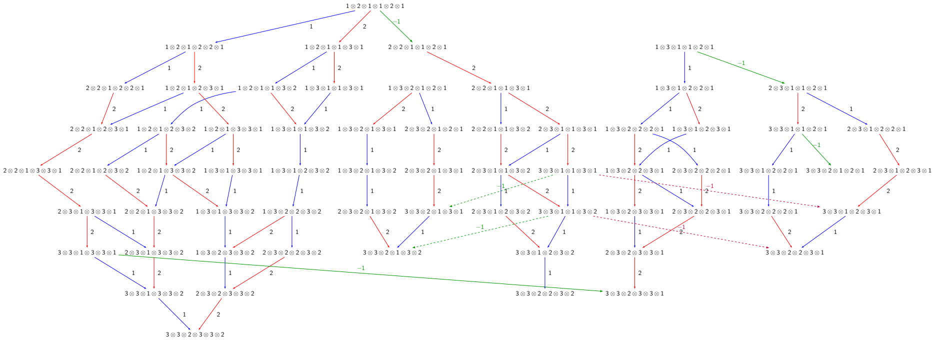
Conjecture (Assaf, Oguz 2018)

In addition to the Stembridge axioms, the relations below uniquely characterize queer crystals.



Counterexample

[Gillespie, Hawkes, Poh, S. 2018]



Graph on type A components

Question: How are the type A components glued together?

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Definition

$\mathcal{C}$ crystal with index set $I_0 \cup \{-1\}$, A_n Stembridge crystal when restricted to I_0

Type A graph $G(\mathcal{C})$ defined as follows:

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Type A graph $G(\mathcal{C})$ defined as follows:

- Vertices of $G(\mathcal{C})$ are the type A components of $\mathcal{C}$, labeled by highest weight elements
- Edge from vertex C_1 to vertex C_2, if $\exists\ b_1 \in C_1$ and $b_2 \in C_2$ such that

$$f_{-1} b_1 = b_2.$$

Graph on type A components: example

correct graph $G(\mathcal{C})$

$$\begin{tikzcd}
& 1 \otimes 2 \otimes 1 \otimes 1 \otimes 2 \otimes 1 \arrow[dl, bend right] \arrow[dr] & \\
2 \otimes 2 \otimes 1 \otimes 1 \otimes 2 \otimes 1 \arrow[d] & & 1 \otimes 3 \otimes 1 \otimes 1 \otimes 2 \otimes 1 \arrow[d] \\
1 \otimes 3 \otimes 2 \otimes 1 \otimes 2 \otimes 1 & & 2 \otimes 3 \otimes 1 \otimes 1 \otimes 2 \otimes 1 \arrow[d] \\
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Combinatorial description of $G(\mathcal{C})$

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- Edges described by e_{-i}:

Proposition (GHPS 2018)

C_1, C_2 distinct type A components in $\mathcal{C}$
Let $u_2 \in C_2$ be I_0-highest weight element

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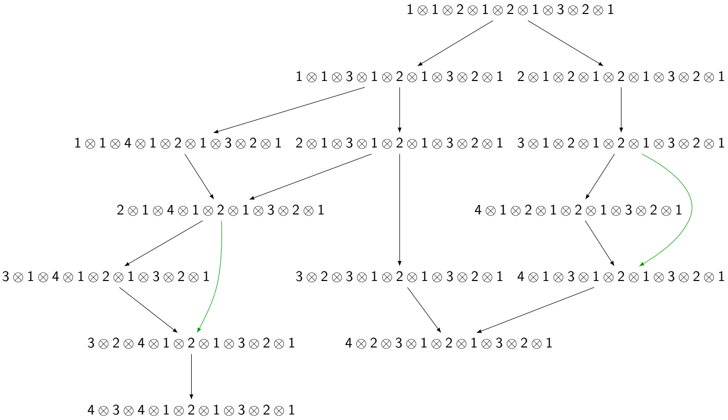
Proposition (GHPS 2018)

C_1, C_2 distinct type A components in $\mathcal{C}$
Let $u_2 \in C_2$ be I_0-highest weight element

There is an edge from C_1 to C_2 in $G(\mathcal{C})$
$\Leftrightarrow e_{-i}u_2 \in C_1$ *for some* $i \in I_0$

Combinatorial description of $G(\mathcal{C})$ (continued)

- Remove by-pass arrows:



Combinatorial description of $G(\mathcal{C})$ (continued)

- Combinatorial description of remaining arrows:
  Define $f_{(-i,h)} := f_{-i} f_{i+1} f_{i+2} \cdots f_{h-1}$.

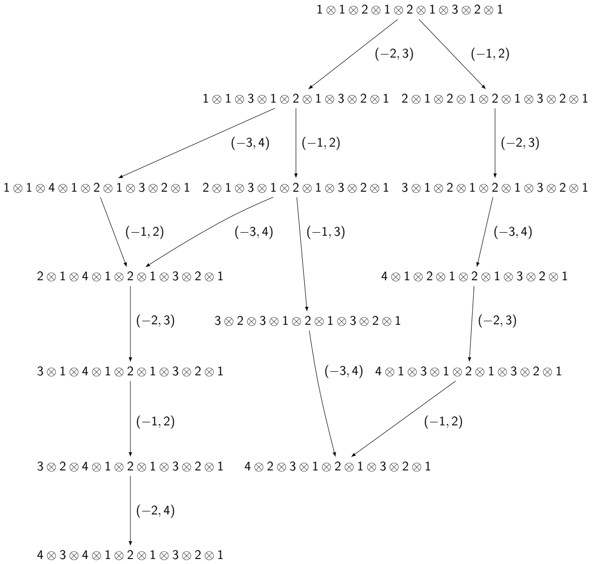
Combinatorial description of $G(\mathcal{C})$ (continued)

- Combinatorial description of remaining arrows:
  Define $f_{(-i,h)} := f_{-i} f_{i+1} f_{i+2} \cdots f_{h-1}$.

Theorem (GHPS 2018)

Let $\mathcal{C}$ be a connected component in $\mathcal{B}^{\otimes \ell}$. Then each non by-pass edge in $G(\mathcal{C})$ can be obtained by $f_{(-i,h)}$ for some i and $h > i$ minimal such that $f_{(-i,h)}$ applies.

Combinatorial description of $G(\mathcal{C})$ (continued)



Combinatorial description of f_{-i}

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$b = 1331242312111$ and $i = 3$

We overline b_{q_j}

$$b = 1\overline{3}31\overline{2}423\overline{1}2111$$

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Here $k = 1$.

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Proposition

Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i\}$-highest weight for $i \in I_0$ and $\varphi_{-i}(b) = 1$. Then $f_{-i}(b)$ is obtained from b by changing

- $b_{q_j} = j$ *to* $j - 1$ *for* $j = i, i-1, \ldots, k+1$
- $b_{r_j} = j$ *to* $j + 1$ *for* $j = i, i-1, \ldots, k$.

Combinatorial description of f_{-i} (continued)

- $b_{q_i}, b_{q_{i-1}}, \ldots, b_{q_1}$ leftmost sequence $i, i-1, \ldots, 1$ from left to right
- Set $r_1 = q_1$
- Recursively $r_j < r_{j-1}$ for $1 < j \leqslant i$ maximal such that $b_{r_j} = j$.
- By definition $q_j \leqslant r_j$. Let $1 \leqslant k \leqslant i$ be maximal such that $q_k = r_k$.

Proposition

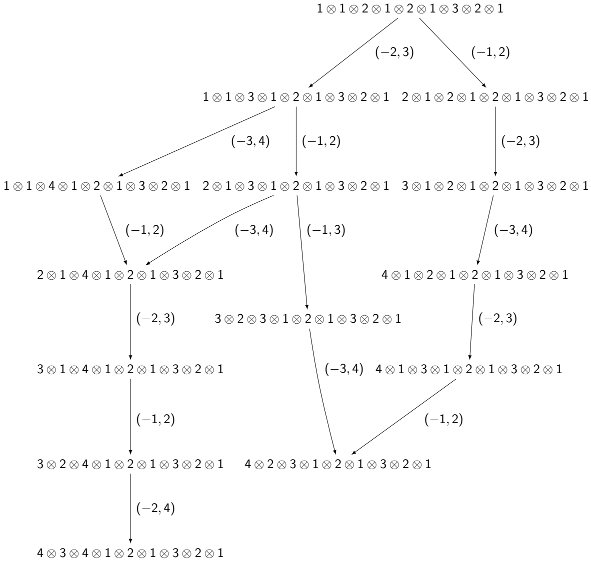
Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i\}$-highest weight for $i \in I_0$ and $\varphi_{-i}(b) = 1$. Then $f_{-i}(b)$ is obtained from b by changing

- $b_{q_j} = j$ *to* $j - 1$ *for* $j = i, i-1, \ldots, k+1$
- $b_{r_j} = j$ *to* $j + 1$ *for* $j = i, i-1, \ldots, k$.

Example

$$b = 1\overline{3}\underline{3}1\overline{2}4\underline{2}3\underline{\overline{1}}2111 \qquad i = 3$$

$$f_{-3}(b) = 124114332 2111$$


1 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1
(−2, 3)
(−1, 2)
1 ⊗ 1 ⊗ 3 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1
2 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1
(−3, 4)
(−1, 2)
(−2, 3)
1 ⊗ 1 ⊗ 4 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1
2 ⊗ 1 ⊗ 3 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1
3 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1
(−1, 2)
(−3, 4)
(−1, 3)
(−3, 4)
2 ⊗ 1 ⊗ 4 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1
4 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1
(−2, 3)
3 ⊗ 2 ⊗ 3 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1
(−2, 3)
3 ⊗ 1 ⊗ 4 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1
(−3, 4)
4 ⊗ 1 ⊗ 3 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1
(−1, 2)
(−1, 2)
3 ⊗ 2 ⊗ 4 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1
4 ⊗ 2 ⊗ 3 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1
(−2, 4)
4 ⊗ 3 ⊗ 4 ⊗ 1 ⊗ 2 ⊗ 1 ⊗ 3 ⊗ 2 ⊗ 1

Thank you !