Lie Superalgebras and Sage

Daniel Bump

July 26, 2018

With the connivance of Brubaker, Schilling and Scrimshaw.

Lie Methods in Sage

Lie methods in WeylCharacter class:

- Compute characters of representations of Lie groups
- Tensor product
- Symmetric and Exterior powers
- Branching Rules
- Functionality is complete and fast

Other relevant tools already in Sage include:

- Crystal bases
- Integrable highest-weight representations of affine Lie algebras
- Symmetric Function code

Lie superalgebras

- In mathematical physics, one encounters symmetries that mix commuting and anticommuting variables.
- Lie superalgebras are a framework for studying these.
- A *super* vector space is a Z_2 graded vector space $V = V_0 \oplus V_1$.
- If $V_0 = \mathbb{C}^m$ and $V_1 = \mathbb{C}^n$ we use the notation $V = \mathbb{C}^{m|n}$.

Example: Let $\bigvee(U)$ and $\bigwedge(U)$ be the symmetric and exterior algebras over a vector space U. If V is a super vector space

$$\bigvee(V) = \bigvee(V_0) \otimes \bigwedge(V_1),$$

$$\bigwedge(V) = \bigwedge(V_0) \otimes \bigvee(V_1).$$

$\mathfrak{gl}(m|n)$

- Many algebraic structures have super analogs.
- $\mathrm{End}(V)$ is itself a super vector space.
- $\mathrm{End}(V)_0 = \mathrm{End}(V_0) \oplus \mathrm{End}(V_1)$.
- $\mathrm{End}(V)_1 = \mathrm{Hom}(V_0, V_1) \oplus \mathrm{Hom}(V_1, V_0)$
- If $V = \mathbb{C}^{m|n}$ then $\mathfrak{gl}(m|n) = \mathrm{End}(V)$

The Lie bracket is modified:

$$[X, Y] = XY - (-1)^{\deg(X)\deg(Y)} YX$$

This illustrates how all algebraic operations are modified in the super world. When two elements of odd degree are interchanged, there is a sign introduced.

Sage considerations

If $\mathfrak{g}$ is a Lie superalgebra then $\mathfrak{g}_0$ is a Lie algebra. Therefore we may inherit from the `WeylCharacterRing` instance for $\mathfrak{g}_0$.

There should be some general code for working with Lie superalgebras, their root systems and characters.

However implementing full-feature code for all Lie superalgebras seems a long range goal.

It may be good to get working code for a few particular Lie superalgebras beginning with $\mathfrak{gl}(m|n)$.

Other Lie superalgebras with high priority are $\mathfrak{osp}$ and $\mathfrak{q}(n)$.

History of $\mathfrak{gl}(m|n)$

- Kac: foundational work, Kac modules
- Berele and Regev: supersymmetric Schur functions, polynomial representations
- Hughes, King, van der Jeugt and Mieg-Thierry: much work culminating in a general (conjectural) formula for irreducible characters; and a rigorous formula for atypicality 1.
- Serganova introduced ideas of Kazhdan-Lusztig theory leading to a satisfactory theory
- Brundan: character formula
- Su and Zhang: character formula

$\mathfrak{gl}(m|n)$

- Let $\mathfrak{g}$ be the Lie superalgebra $\mathfrak{gl}(m|n)$.
- Let $\mathfrak{h}$ denote the diagonal (Cartan) subalgebra of $\mathfrak{g}$.
- The weight lattice $\Lambda \cong \mathbb{Z}^{m+n}$ of $\mathfrak{g}$ may be identified with the weight lattice of its even part $\mathfrak{g}_0 = \mathfrak{gl}(m) \times \mathfrak{gl}(n)$.
- The lattice Λ comes with an invariant bilinear form $(\lambda|\mu)$ of signature (m, n).
- If $\{\mathbf{e}\}_{i=1}^{m+n}$ is the standard basis vectors of Λ, then

$$(\mathbf{e}_i|\mathbf{e}_j) = \begin{cases} 1 & i \leqslant m \\ -1 & i > m \end{cases}$$

Root system

- The root system $\Phi = \Phi_0 \cup \Phi_1$, where Φ_0 (resp. Φ_1) is the set of even (respectively odd) roots.
- If $\mathbf{e}_i$ ($1 \leqslant i \leqslant m+n$) are the standard basis vectors, then the positive roots consist of $\alpha_{ij} = \mathbf{e}_i - \mathbf{e}_j$ with $1 \leqslant i < j \leqslant m+n$.
- The odd positive roots α_{ij} with $1 \leqslant i \leqslant m$, $m+1 \leqslant j \leqslant m+n$ are all isotropic.

$$\left(\begin{array}{|c|c|} \hline \text{even} & \text{odd} \\ \hline \text{odd} & \text{even} \\ \hline \end{array}\right)$$

Atypicality

A weight $\lambda = (\lambda_1, \ldots, \lambda_{m+n})$ is dominant if $\lambda_1 \leqslant \cdots \leqslant \lambda_m$ and $\lambda_{m+1} \leqslant \cdots \leqslant \lambda_{m+n}$.

Kac defined the notion of atypicality of the dominant weight λ to be the number of odd positive roots α such that $(\lambda + \rho|\alpha) = 0$.

We say such roots α are atypical for λ.

If the atypicality is 0, we call λ typical. For these the representation theory is simple.

Atypicality 1 starts to show interesting behavior but is still not too hard.

Representations

Every dominant weight λ parametrizes an indecomposable Kac module $K(\lambda) = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0 \oplus \mathfrak{u}_1^+} V_0(\lambda)$.

Here $V_0(\lambda)$ is the unique irreducible module of $\mathfrak{g}_0$ with highest weight λ, and $\mathfrak{u}_1^+$ is the abelian subalgebra generated by the odd positive root spaces.

There is also a unique irreducible module $L(\lambda)$ with highest weight λ, which is the unique irreducible quotient of $K(\lambda)$.

$K(\lambda)$ has a nice character formula.

If λ is typical then $K(\lambda) = L(\lambda)$. In general the character of $L(\lambda)$ is harder to compute.

Characters of Kac modules

The character $\chi_{K(\lambda)}$ of the Kac module has a simple description. Let

$$L_0 = \prod_{\alpha\in\Phi_0^+} (e^{\alpha/2} - e^{-\alpha/2}), \qquad L_1 = \prod_{\alpha\in\Phi_1^+} (e^{\alpha/2} + e^{-\alpha/2}).$$

Let $W = S_m \times S_n$ (Weyl group), $\rho = \rho_0 - \rho_1$ Where ρ_0 (resp. ρ_1) is half the sum of the even (resp. odd) positive roots. Then

$$\mathrm{ch}_{K(\lambda)} = \frac{L_1}{L_0} \sum_{w\in W} \varepsilon(w)\, e^{w(\lambda+\rho)}.$$

Characters of Kac modules (continued)

This can be written:

$$L_0^{-1} \sum_{w \in W} \varepsilon(w)\, w \left(\left(\prod_{\alpha \in \Phi_1^+} (1 + e^{-\alpha}) \right) e^{\lambda + \rho_0} \right).$$

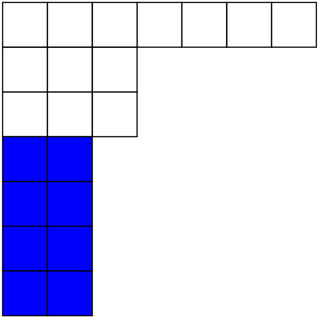
Expanding the product, this can be evaluated using the Weyl character formula for $\mathfrak{g}_0$. So Kac modules have nice character formulas.

Polynomial representations

There are two (overlapping but distinct) classes of irreducibles for which there is a nice character formula.

If λ is a (m, n) hook partition whose Young diagram omits the box $(m+1, n+1)$ then there is a dominant weight λ^* obtained by transposing part of λ. Example: $m = n = 3$

$\lambda =$



$$\lambda^* = (7, 3, 3; 4, 4)$$

Polynomial representations (continued)

In this case Berele and Regev showed that the character of $L(\lambda^*)$ is the supersymmetric Schur function $s_\lambda(t|u)$.

$$s_\lambda(t|u) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu(t) s_{\nu'}(u)$$

where $c^\lambda_{\mu,\nu}$ is the Littlewood-Richardson coefficient.

A different class of irreducibles with nice characters are $L(\lambda)$ where λ is typical. In this case $L(\lambda) = K(\lambda)$ and we have already seen the character formula.

Atypicality one

Theorem (Hughes, King, van der Jeugt and Thierry-Mieg)

If λ has atypicality 1, then $K(\lambda)$ has length 2: there is a short exact sequence

$$0 \longrightarrow L(\mu) \longrightarrow K(\lambda) \longrightarrow L(\lambda) \longrightarrow 0,$$

where $L(\mu)$ is another irreducible module. The dominant weight μ also has atypicality 1. Let α be the atypical root, i.e. the unique $\alpha \in \Phi_1^+$ with $(\alpha|\lambda+\rho) = 0$. Then

$$\chi_{L(\lambda)} = L_0^{-1} \sum_{w \in W} \varepsilon(w)\, w \left(\left(\prod_{\substack{\gamma \in \Phi_1^+ \\ \gamma \neq \alpha}} (1 + e^{-\alpha\gamma}) \right) e^{\lambda+\rho_0} \right)$$

Sage implementation

The character formulas for Kac modules and for irreducibles with atypicality 0 and 1 are implemented in some preliminary code.

This code is not polished and not merged in Sage. But it works.

You can find the file `combinat/crystals/scharacter.sage` in the branch `public/stensor`.

The SuperWeylCharacterRing class inherits from WeylCharacterRing.

It is desirable to remove the limitation on atypicality.

Crystals

Two classes of $\mathfrak{gl}(m|n)$ modules have nice crystal bases.

- Polynomial representations (Benkart, Kang and Kashiwara)
- Kac crystals (Jae-Hoon Kwon)

Thanks to Franco Saliola, Travis Scrimshaw and Anne Schilling, these are implemented in Sage.
Both these theories are rooted in the theory of quantum groups.

Crystals of atypicality 1

Crystal bases of modules of atypicality 0 are known thanks to Kwon, since in this case $L(\lambda) = K(\lambda)$.

For atypicality 1, recall that we have a short exact sequence

$$0 \longrightarrow L(\mu) \longrightarrow K(\lambda) \longrightarrow L(\lambda) \longrightarrow 0,$$

In particularly favorable cases, one of $L(\mu)$ or $L(\lambda)$ might be polynomial and the other not.

Say $L(\lambda)$ is polynomial.

In this case, we think a crystal base for $L(\mu)$ can be concocted by identifying the crystal for $L(\lambda)$ inside of $K(\lambda)$ and discarding it. More generally, crystals of atypicality 1 can be sought by a procedure of cutting apart Kac crystals.

Cutting the Kac crystal

A first idea is that one eliminates 0 arrows from the crystal if the head v of the arrow has $(\mathrm{wt}(v), h_0) = 0$.

This procedure (slightly modified) seems to work in practice, but it is a farther step removed from the origins of crystal bases in the theory of quantum groups. It is not certain that a nice theory exists.

The definitions followed by BKK and Kwon will require modification before they can be used in atypicality 1.

These experiments may point the way to a solution to this problem.