Algorithms for Cox rings

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ICERM
May 2018

Cox rings

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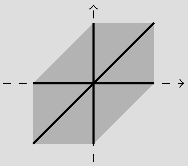
with $\deg(T_\varrho) = [D_\varrho] \in \mathrm{Cl}(X)$.



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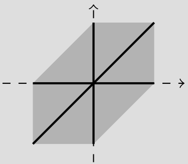

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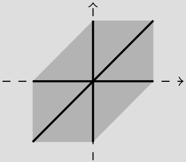

Features: significant invariant, $\mathrm{Cl}(X)$-factorial.

Mori dream spaces

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Example

The class of Mori dream spaces comprises

- toric varieties, spherical varieties,
- rational complexity-one T-varieties,
- smooth Fano varieties,
- general hypersurfaces in $\mathbb{P}_n$, $n \geq 4$.

Mori dream spaces: combinatorial description

Explicit description (Berchtold/Hausen)

$$\left\{\begin{array}{c}\text{Mori dream}\\ \text{spaces}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{factorially } K\text{-graded}\\ \text{algebras } R \text{ with a}\\ \text{vector in } \mathrm{Mov}(R)\end{array}\right\}$$

$$X \quad \mapsto \quad (\mathrm{Cox}(X), \mathrm{Cl}(X), \text{ample class})$$

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$$\begin{array}{rcl} X & \mapsto & (\mathrm{Cox}(X), \mathrm{Cl}(X), \text{ample class}) \\ (\mathrm{Spec}\, R)^{\mathrm{ss}}(w) /\!\!/ \mathrm{Spec}\, \mathbb{C}[K] & \leftarrow\!\shortmid & (R, K, w) \end{array}$$

Remark:

- the vector w fixes a *GIT-cone*,
- this allows a treatment of Mori dream spaces in terms of commutative algebra and polyhedral combinatorics.

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Fan Σ_X of X:

$$0 \longleftarrow K \xleftarrow[\deg(T_i) \,\leftarrow\!\shortmid\, e_i]{Q} \mathbb{Q}^r \longleftarrow M_{\mathbb{Q}} \longleftarrow 0$$

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Then Σ_X is the normalfan over the *fiber polytope*

$$B_w := Q^{-1}(w) \cap \mathbb{Q}^r_{\geq 0} - w' \subseteq \ker(Q) \cong M_{\mathbb{Q}}.$$

Mori Dream Spaces: computational approach

Mori Dream Spaces: computer algebra approach

Aim

Let X be a Mori dream space.

1. Given $(\mathrm{Cox}(X), \mathrm{Cl}(X), w)$, explore the geometry of X computationally.

Mori Dream Spaces: computer algebra approach

Aim

Let X be a Mori dream space.

1. Given $(\mathrm{Cox}(X), \mathrm{Cl}(X), w)$, explore the geometry of X computationally.
2. Given X, compute its defining data $(\mathrm{Cox}(X), \mathrm{Cl}(X), w)$.

MDSpackage

Basic algorithms for Mori dream spaces implemented in `MDSpackage` (for `Maple`, with Hausen, LMS J. COMPUT. MATH.).

`MDSpackage`: basic algorithms

For general Mori dream spaces:

- Basics on K-graded algebras,
- Picard group, cones of divisor classes,
- canonical toric ambient variety,
- singularities,
- test for being factorial, ...

`MDSpackage`: basic algorithms

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For complete intersections:

- intersection numbers,
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For complete intersections:

- intersection numbers,
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For complexity-one T-varieties:

- roots of the automorphism group,
- test for being (ε-log) terminal, ...

MDSpackage: examples

Example (Data fixing a Mori dream space)

1. Define the Cox ring

$$\mathrm{Cox}(X) \;:=\; \mathbb{C}[T_1, \ldots, T_8]/\langle T_1 T_6 + T_2 T_5 + T_3 T_4 + T_7 T_8 \rangle,$$

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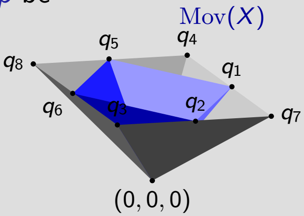
3. Set $q_i := \deg(T_i)$. Let the *degree map* be

$$Q \ := \ [q_1, \ldots, q_8] \ := \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 2 & -2 \\ 0 & 1 & 1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}.$$



`MDSpackage`: examples

Example (continued)

After having entered X in `MDSpackage`:

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> MDSpic(X);
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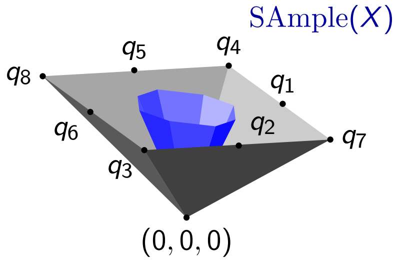

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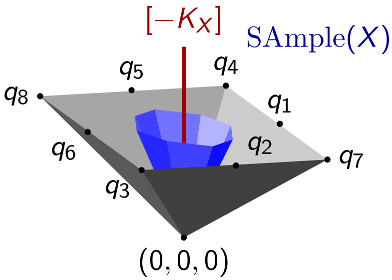

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Computing the Mori chamber decomposition

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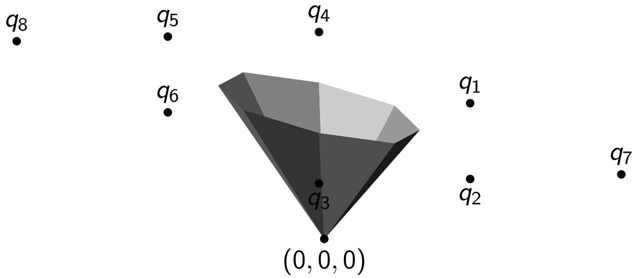

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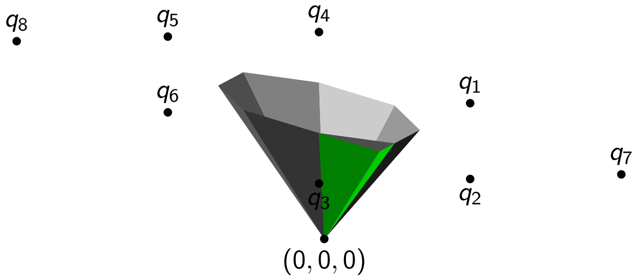

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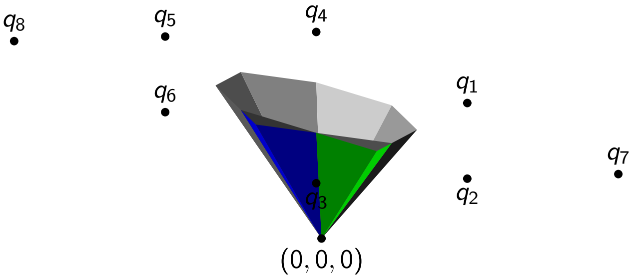

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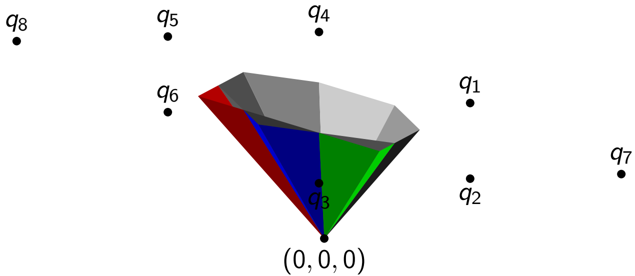

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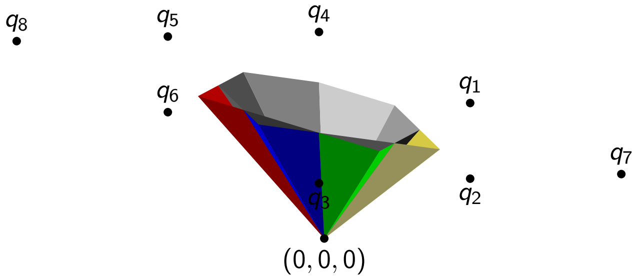

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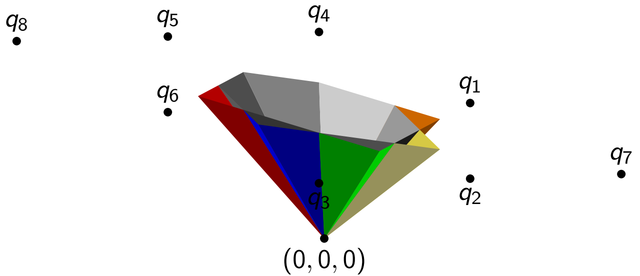

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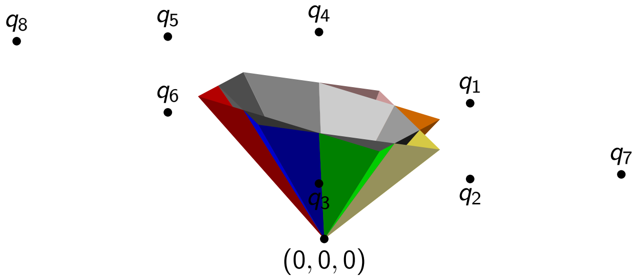

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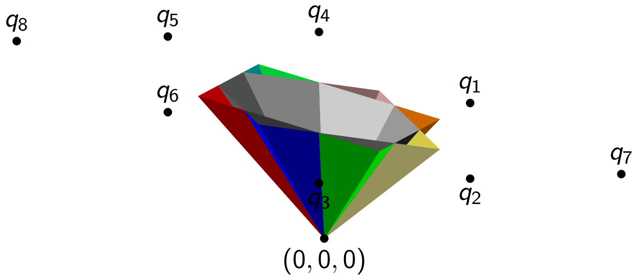

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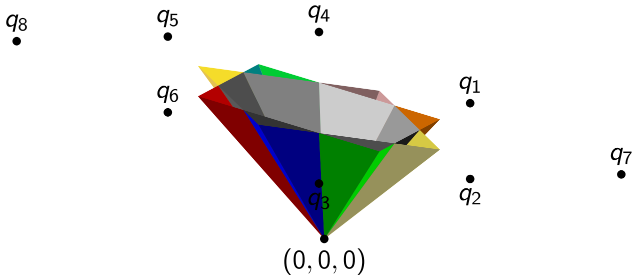

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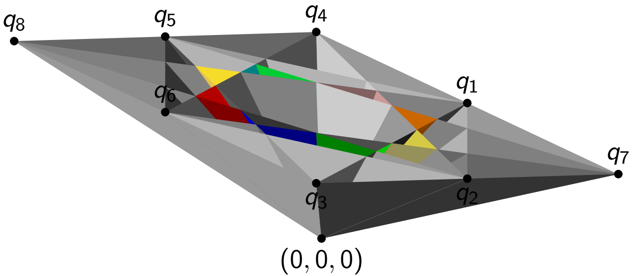

Advanced algorithms

We present these recent, advanced algorithms:

1. Compute Cox rings of blow ups (with Hausen, Laface)
2. Compute symmetries of Mori dream spaces (with Hausen, Wolf).

(1) Computing Cox rings of modifications

Cox rings of blow ups

Aim: Given a blow up $X_2 \to X_1$ of a Mori dream space X_1, compute $\mathrm{Cox}(X_2)$ if finitely generated.

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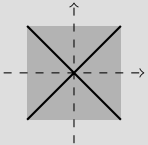
Example

Let X_2 be the blow up of a general point $p = [1,1,1,1]$ of

$$X_1 := \mathrm{ToricVariety}\left(\quad \right)$$



What is $\mathrm{Cox}(X_2)$?

Cox rings of blow ups (with Hausen, Laface)

- Blow up along $C \subseteq X_1$ irreducible subvariety, $C \subseteq X_1^{\mathrm{reg}}$.

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Proposition: $\mathrm{Cox}(X_2)$ is isomorphic to the *saturated Rees algebra*

$$\bigoplus_{k \in \mathbb{Z}} \left(I^{-k} : J^{\infty} \right) t^k \qquad \text{where } J \text{ is the irrelevant ideal.}$$

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Algorithm

Input: X_1 and $I \subseteq \mathrm{Cox}(X_1)$.
Output: $\mathrm{Cox}(X_2)$ if and only if X_2 is a Mori dream space.

for each $n = 1, 2, \ldots$ **do**
  if $\mathrm{Cox}(X_2)$ is generated in the Rees-components $(I^{-k} : J^{\infty})$ with $-k \leq n$:
    return an explicit description of $\mathrm{Cox}(X_2)$.

Cox rings of blow ups

Example (continued)

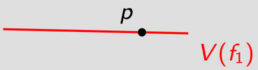
1. the ideal of $C := \{p\} \subseteq X_1$ is generated by

$$
\begin{aligned}
f_1 &:= T_2 T_3 - T_1 T_4\,, \\
f_2 &:= T_1 T_2 - T_3 T_4\,, \\
f_3 &:= T_1^2 - T_3^2, \\
f_4 &:= T_2^2 - T_4^2 \\
&\in \operatorname{Cox}(X_1) = \mathbb{C}[T_1, \ldots, T_4].
\end{aligned}
$$

Cox rings of blow ups

Example (continued)

❶ the ideal of $C := \{p\} \subseteq X_1$ is generated by



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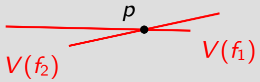

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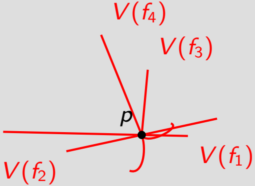

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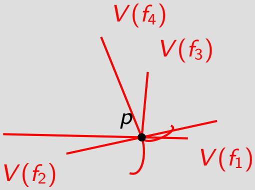

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&\quad \in \mathrm{Cox}(X_1) = \mathbb{C}[T_1, \ldots, T_4].
\end{aligned}
$$

2. This means to embed the total coordinate space

$$\overline{X_1} = \mathbb{C}^4 \xrightarrow{\;x \mapsto (x, f_1(x), \ldots, f_4(x))\;} \mathbb{C}^8$$

and we identify $\overline{X}_1 = \mathbb{C}^4$ with $V(T_{4+i} - f_i) \subseteq \mathbb{C}^8$.

Cox rings of blow ups

Example (continued)

❸ **Toric ambient modification:** blow up $\overline{\mathbb{T}_{Z_1} \cdot p} \subseteq Z_1$:

$$
\begin{array}{ccccc}
 & & X_1 & \subseteq & Z_1 \\
 & \nearrow & & & & \nwarrow \\
V(T_{4+i} - f_i) = \overline{X_1} & & & \subseteq & & & \mathbb{C}^8
\end{array}
$$

Cox rings of blow ups

Example (continued)

❸ **Toric ambient modification:** blow up $\overline{\mathbb{T}_{Z_1} \cdot p} \subseteq Z_1$:

$$\begin{array}{ccccc} & & X_2 & \subseteq & Z_2 \\ & & \downarrow{\scriptstyle \pi} & & {\scriptstyle \text{toric}}\downarrow{\scriptstyle \pi} \\ & & X_1 & \subseteq & Z_1 \\ & \nearrow & & & & \nwarrow \\ V(T_{4+i} - f_i) = \overline{X_1} & & & \subseteq & & & \mathbb{C}^8 \end{array}$$

Cox rings of blow ups

Example (continued)

❸ **Toric ambient modification:** blow up $\overline{\mathbb{T}_{Z_1} \cdot p} \subseteq Z_1$:

$$
\begin{tikzcd}
\overline{\overline{\pi}^{-1}(\overline{X_1} \cap (\mathbb{C}^*)^8)} = \overline{X_2} \arrow[rrr, phantom, "\subseteq"] \arrow[dddd, "\overline{\pi}"'] \arrow[ddr, dashed] & & & \mathbb{C}^9 \arrow[ddl, dashed] \arrow[dddd, "\overline{\pi}"] \\
& & & \\
& X_2 \arrow[r, phantom, "\subseteq"] \arrow[d, "\pi"'] & Z_2 \arrow[d, "\pi", "\text{toric}"'] & \\
& X_1 \arrow[r, phantom, "\subseteq"] & Z_1 & \\
V(T_{4+i} - f_i) = \overline{X_1} \arrow[rrr, phantom, "\subseteq"] \arrow[ur, dashed] & & & \mathbb{C}^8 \arrow[ul, dashed]
\end{tikzcd}
$$

Cox rings of blow ups

Example (continued)

4. **In terms of Cox rings:** Let $I_1 := \langle T_{4+i} - f_i \rangle$. We obtain $I_2 \leq \mathbb{C}[T_1, \ldots, T_9]$ via

$$I_1 \quad \leq \quad \mathrm{Cox}(Z_1) = \mathbb{C}[T_1, \ldots, T_8] \longrightarrow \mathbb{C}[T_1^{\pm 1}, \ldots, T_8^{\pm 1}]$$

Cox rings of blow ups

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I_1 & \leq & \mathrm{Cox}(Z_1) \arrow[r, equal] & \mathbb{C}[T_1, \ldots, T_8] \arrow[r] & \mathbb{C}[T_1^{\pm 1}, \ldots, T_8^{\pm 1}]
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Cox rings of blow ups

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\end{tikzcd}
$$

Example: blow up of a Mori dream space

Example (continued)

Then
$$
\begin{aligned}
\mathrm{Cox}(X_2) = \mathbb{C}[T_1, \ldots, T_9]/\langle\, & T_4 T_5 - T_1 T_6 + T_2 T_7\,, \\
& T_3 T_5 - T_1 T_7 + T_2 T_8\,, \\
& T_2 T_5 - T_3 T_6 + T_4 T_7\,, \\
& T_1 T_5 - T_3 T_7 + T_4 T_8\,, \\
& T_2 T_3 - T_1 T_4 - T_5 T_9\,, \\
& T_1 T_2 - T_3 T_4 - T_7 T_9\,, \\
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and X_2 is the *Cayley cubic* $V(wxy + xyz + yzw + zwx) \subseteq \mathbb{P}_3$.

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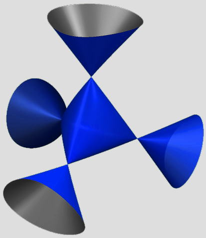
$$



and X_2 is the *Cayley cubic* $V(wxy + xyz + yzw + zwx) \subseteq \mathbb{P}_3$.

Implementation

Our algorithms are implemented in the library `compcox.lib` for the open source algebra system `Singular`.

Applications

We have computed Cox rings of:

1. with Hausen, Laface:
   - Gorenstein log-terminal del Pezzo surfaces X with $\varrho(X) = 1$,
   - smooth rational surfaces with $\varrho(X) \leq 6$,
   - blow ups of $\mathbb{P}_3$.
2. with Derenthal, Hausen, Heim, Laface:
   - smooth non-toric Fano threefolds with $\varrho(X) \leq 2$,
   - cubic surfaces with at most ADE singularities.
3. with Hausen, Laface:
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3. with Hausen, Laface:
   - *study of Cox rings of blow ups of $\mathbb{P}(a, b, c)$* $\rightsquigarrow$ *next slide*

Application: blow ups of $\mathbb{P}(a, b, c)$

Let $X \to \mathbb{P}(a, b, c)$ be the blow up at the point $[1, 1, 1]$.

Theorem (with Hausen, Laface)

Equivalent:

1. *X admits a nontrivial $\mathbb{C}^*$-action.*
2. *$a = mb + nc$ with $m, n \in \mathbb{Z}_{\geq 0}$.*
3. *$\mathrm{Cox}(X)$ generated by elements of Rees degree ≤ 1.*

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In this case: X *is a Mori dream surface with Cox ring*

$$\mathcal{R}(X) \ = \ \mathbb{C}[T_1, \ldots, T_5]/\langle T_3 T_4 - T_1^c + T_2^b \rangle,$$

$$Q \ = \ \begin{bmatrix} b & c & bc & 0 & a \\ 0 & 0 & -1 & 1 & -1 \end{bmatrix}.$$

Application: blow ups of $\mathbb{P}(a, b, c)$

Let $X \to \mathbb{P}(a, b, c)$ be the blow up at the point $[1, 1, 1]$.

Theorem (with Hausen, Laface)

Assume X is not a $\mathbb{C}^*$-surface. Equivalent:

1. $\operatorname{Cox}(X)$ *generated by elements of Rees degree* ≤ 2.
2. *After reordering:* $2a = nb + mc$ *with* $n, m \in \mathbb{Z}_{\geq 0}$ *such that* $b \geq 3m$ *and* $c \geq 3n$.

In this case, X is a Mori dream surface with Cox ring

$$
\begin{aligned}
\mathcal{R}(X) &= \mathbb{C}[x, y, z, t_1, \ldots, t_4, s]/(I_2 : s^\infty), \\
Q &= \begin{bmatrix} a & b & c & 2a & \frac{b(c+n)}{2} & \frac{c(b+m)}{2} & bc & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & -2 & 1 \end{bmatrix},
\end{aligned}
$$

where ...

Application: blow ups of $\mathbb{P}(a, b, c)$

Theorem (continued)

where $I_2 \subseteq \mathbb{C}[x, y, z, t_1, \dots, t_4, s]$ *is generated by*

$$x^2 - y^n z^m - t_1 s, \qquad xz^{\frac{b-m}{2}} - y^{\frac{c+n}{2}} - t_2 s,$$
$$xy^{\frac{c-n}{2}} - z^{\frac{b+m}{2}} - t_3 s,$$
$$xy^{\frac{c-3n}{2}} z^{\frac{b-3m}{2}} t_1 - y^{\frac{c-n}{2}} t_2 - z^{\frac{b-m}{2}} t_3 - t_4 s,$$
$$y^{\frac{c-3n}{2}} z^{\frac{b-3m}{2}} t_1^2 - t_2 t_3 - x t_4,$$
$$y^{\frac{c-n}{2}} t_1 - z^m t_2 - x t_3,$$
$$z^{\frac{b-m}{2}} t_1 - x t_2 - y^n t_3,$$
$$t_3^2 + y^{\frac{c-3n}{2}} t_1 t_2 - z^m t_4,$$
$$t_2^2 + z^{\frac{b-3m}{2}} t_1 t_3 - y t_4.$$

(2) Computing symmetries of graded algebras and MDSs

Symmetries of graded algebras

Setting: Consider an affine, integral $\mathbb{C}$-algebra

$$R \;=\; \mathbb{C}[T_1,\dots,T_r]/I \;=\; \bigoplus_{w\in K} R_w$$

graded *pointedly* by a finitely generated abelian group K,

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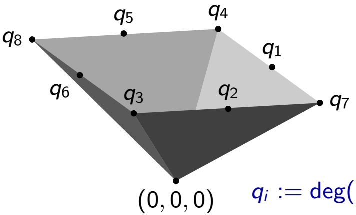

$q_i := \deg(T_i)$

Symmetries of graded algebras

Example (continued)

The $\mathbb{C}$-algebra $R := \mathbb{C}[T_1, \ldots, T_5]/\langle T_1 T_2 + T_3^2 + T_4^2 \rangle$ is pointedly $K := \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$-graded via

Symmetries of graded algebras

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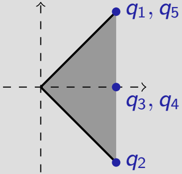
$$[q_1, \ldots, q_5] := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix},$$

$$q_i := \deg(T_i) \in K.$$



$\subseteq K \otimes \mathbb{Q} = \mathbb{Q}^2$

Symmetries of graded algebras

The *automorphism group* $\mathrm{Aut}_K(R)$ of a K-graded algebra R consists of all pairs (φ, ψ) with

- $\varphi\colon R \to R$ automorphism of $\mathbb{C}$-algebras,
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Aim

Compute $\mathrm{Aut}_K(R)$ as $V(J) \subseteq \mathrm{GL}(n)$ for some n.

Symmetries of graded algebras (with Hausen, Wolf)

Proposition

❶ *Write* $G := \mathrm{Aut}_K(\mathbb{C}[T_1, \ldots, T_r])$*. There is an isomorphism*

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finite problem

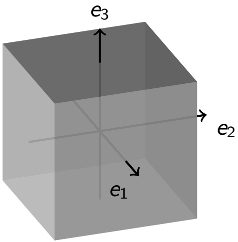
Symmetries of graded algebras (with Hausen, Wolf)

Algorithm (compute $\mathrm{Aut}_K(R)$)

1. Represent $\mathrm{Aut}_K(\mathbb{C}[T_1, \ldots, T_r])$ as a subgroup $G \subseteq \mathrm{GL}(n, \mathbb{C})$.
2. For $(\varphi, \psi) \in G$, the condition $(\varphi, \psi) \cdot I_{q_i} = I_{\psi(q_i)}$ yields $J \subseteq \mathcal{O}(G)$ with

$$\underbrace{\mathrm{Stab}_I(G)}_{\mathrm{Aut}_K(R)} \;=\; V(J) \;\subseteq\; G.$$

Task: compute lattice points



Symmetries of graded algebras (with Hausen, Wolf)

Algorithm (compute $\mathrm{Aut}_K(R)$)

1. Represent $\mathrm{Aut}_K(\mathbb{C}[T_1, \dots, T_r])$ as a subgroup $G \subseteq \mathrm{GL}(n, \mathbb{C})$.
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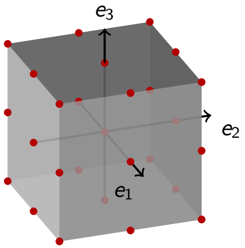

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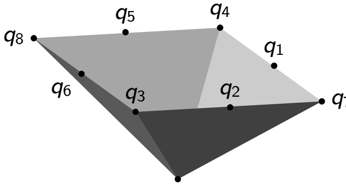

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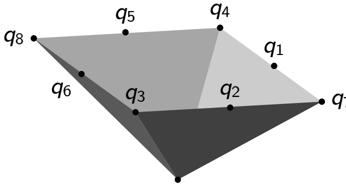

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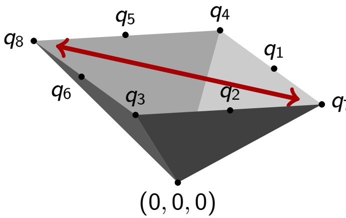

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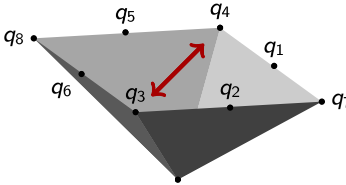

Graded algebras: symmetries

Example (continued)

The group $\mathrm{Aut}_K(R)$ is isomorphic to the following subgroup of $\mathrm{GL}(5, \mathbb{C})$:

$$\left\{ \begin{bmatrix} Y_1 & 0 & 0 & 0 & 0 \\ 0 & Y_7 & 0 & 0 & 0 \\ 0 & 0 & Y_{13} & 0 & 0 \\ 0 & 0 & 0 & Y_{19} & 0 \\ 0 & 0 & 0 & 0 & Y_{25} \end{bmatrix} \in \mathrm{GL}(5, \mathbb{C}); \quad \begin{array}{l} Y_{13}^2 = Y_{19}^2, \\ Y_1 Y_7 = Y_{13}^2 \end{array} \right\}$$

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In particular:

$$\dim(\mathrm{Aut}_K(R)) = 3, \qquad \sharp\ \text{components} = 4.$$

Symmetries of Mori dream spaces

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Theorem (Arzhantsev/Hausen/Huggenberger/Liendo)

Exact sequence of linear algebraic groups:

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$$\begin{tikzcd}
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 & & \mathrm{Aut}_H\left(\overline{X}\right) \arrow[d, phantom, "\subseteq" rotate=90] & & \\
1 \arrow[r] & H \arrow[r] & \boxed{\mathrm{Aut}_H(\widehat{X})} \arrow[r] & \boxed{\mathrm{Aut}(X)} \arrow[r] & 1
\end{tikzcd}$$

$\rightsquigarrow$ *fix* $\lambda(w)$ $\quad$ $\rightsquigarrow$ *hilbert basis*

Applications and software

Implemented in `autgradalg.lib` for `Singular`.

Applications:

- with Hausen, Wolf:
  - $\mathrm{Aut}(X)$ for singular cubic surfaces with at most ADE singularities,
  - compute symmetries of homogeneous ideals.
- Preprint:
  - Certain non-toric terminal Fano threefold of Picard number one with an effective two-torus action.