Rationality constructions for cubic hypersurfaces

ICERM workshop 'Birational Geometry and Arithmetic'

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May 14, 2018

Goals for this talk

Our focus is smooth cubic fourfolds $X \subset \mathbb{P}^5$:

1. Review recent progress on rationality
2. Place these results in the larger conjectural context
3. Propose next steps for future work

The more recent results I will present are joint with Addington, Tschinkel and Várilly-Alvarado, along with recent work of Kuan-Wen Lai.

Classical rational parametrizations

Cubic fourfolds containing planes

Consider a cubic fourfolds containing two disjoint planes

$$P_1, P_2 \subset X, \quad P_i \simeq \mathbb{P}^2.$$

The 'third-point' construction

$$\begin{array}{cccc} \rho : P_1 \times P_2 & \overset{\sim}{\dashrightarrow} & X \\ (p_1, p_2) & \mapsto & x \end{array}$$

is birational, where the line

$$\ell(p_1, p_2) \cap X = \{p_1, p_2, x\}.$$

Writing

$$P_1 = \{u = v = w = 0\} \quad P_2 = \{x = y = z = 0\}$$

then we have

$$X = \{F_{1,2}(u, v, w; x, y, z) + F_{2,1}(u, v, w; x, y, z) = 0\},$$

forms of bidegrees $(1, 2)$ and $(2, 1)$. The indeterminacy of ρ is the locus

$$S = \{F_{1,2} = F_{2,1} = 0\} \subset P_1 \times P_2 \subset \mathbb{P}^8,$$

a K3 surface parametrizing lines in X meeting P_1 and P_2. These are blown down by ρ^{-1}.

Cubic fourfolds containing quartic scrolls

This example is due to Morin-Fano (1940) and Beauville-Donagi (1985).
A quartic scroll is a smooth surface

$$T_4 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^5$$

embedded via forms of bidegree $(1,2)$. The linear system of quadrics cutting out T_4 collapses all its secant lines, inducing a map

$$\mathbb{P}^5 \dashrightarrow Q \subset \mathbb{P}^5$$

onto a hypersurface of degree two. Any cubic fourfold

$$X \supset T_4$$

is mapped birationally to Q and thus is rational.

What is the parametrizing map

$$\rho : Q \overset{\sim}{\dashrightarrow} X?$$

Fix a point on a degree 14 K3 surface

$$s \in S \subset \mathbb{P}^8$$

and take a double (tangential) projection of $\mathrm{Bl}_s(S) \subset \mathbb{P}^5$. The resulting surface is contained in a quadric hypersurface Q and ρ arises from the cubics containing this surface.
Again, we have a K3 surface.

Cubic fourfolds with double point

A cubic fourfold with double point

$$x_0 = [1,0,0,0,0,0] \in X \subset \mathbb{P}^5$$

is always rational via projection from x_0

$$X \overset{\sim}{\dashrightarrow} \mathbb{P}^4.$$

The inverse map ρ blows up a K3 surface

$$S = \{F_2(v,w,x,y,z) = F_3(v,w,x,y,z) = 0\}$$

where $X = \{uF_2 + F_3 = 0\}$.

Classification and conjectures

Moduli space

Let $\mathcal{C}$ denote the moduli space of cubic fourfolds, smooth (as a stack) of dimension 20. The middle Hodge numbers are

$$0 \quad 1 \quad 21 \quad 1 \quad 0.$$

Voisin has shown that the period map for cubic fourfolds is an open immersion into its period domain, a type IV Hermitian symmetric domain – analogous to K3 surfaces. When X is a very general cubic fourfold we have

$$H^{2,2}(X) \cap H^4(X, \mathbb{Z}) = \mathbb{Z}h^2$$

where h is the hyperplane class. Cubic fourfolds with

$$H^{2,2}(X) \cap H^4(X, \mathbb{Z}) \supsetneq \mathbb{Z}h^2$$

are *special*.

Speciality Conjecture

Conjecture (Harris-Mazur ??)
All rational cubic fourfolds are special.

The special cubic fourfolds form a countably infinite union of irreducible divisors

$$\cup_d \mathcal{C}_d \subset \mathcal{C}$$

where $d \equiv 0, 2 \pmod 6$ and $d \geq 8$, e.g.,

- $d = 8$: $X \supset P$ a plane;
- $d = 14$: $X \supset T_4$ a quartic scroll.

While no cubic fourfolds are *known* to be irrational most people doubt that *all* special cubic fourfolds are rational. I would personally be very surprised if the examples

- $d = 12$: $X \supset T_3 \simeq \mathbb{F}_1$ a cubic scroll;
- $d = 20$: $X \supset V \simeq \mathbb{P}^2$ a Veronese surface;

were generally rational. Hence we narrow the search.

All known rational parametrization $\rho : \mathbb{P}^4 \dashrightarrow X$ blow up a K3 surface.

Cubic fourfolds and K3 surfaces

On blowing up a smooth surface S in a fourfold Y, we have

$$H^4(\mathrm{Bl}_S(Y), \mathbb{Z}) = H^4(Y, \mathbb{Z}) \oplus H^2(S, \mathbb{Z})(-1)$$

where the (-1) reflects Tate twist. This motivates the following:

Definition

A polarized K3 surface (S, f) is associated *with a cubic fourfold X if we have a saturated embedding of the primitive Hodge structure*

$$H^2(S, \mathbb{Z})_\circ(-1) \hookrightarrow H^4(X, \mathbb{Z}).$$

It follows that X is special.

Some basic properties:

- a general cubic fourfold $[X] \in \mathcal{C}_d$ admits an associated K3 surface unless $4|d, 9|d$, or $p|d$ for some odd prime $p \equiv 2 \pmod 3$;
- all known rational cubic fourfolds admit associated K3 surfaces;
- Kuznetsov proposed an alternate formulations via derived categories of coherent sheaves – Addington and Thomas have shown this is equivalent to the Hodge characterization over dense open subsets of each $\mathcal{C}_d$;
- distinct polarized K3 surfaces (S_1, f_1) and (S_2, f_2) may have isomorphic primitive cohomologies – this characterizes derived equivalence among rank one K3 surfaces.

A curiosity

Thus associated K3 surfaces are far from unique; the monodromy representation over $\mathcal{C}_d$ when $3|d$ precludes a well-defined choice!
Is there a diagram

$$\begin{array}{ccccc} & & X & & \\ & \stackrel{\beta_1}{\swarrow} & & \stackrel{\beta_2}{\searrow} & \\ \mathbb{P}^4 & & & & \mathbb{P}^4 \end{array}$$

where X is a cubic fourfold, β_i blows up a K3 surface S_i, but S_1 and S_2 are distinct? We would expect the K3 surfaces to be derived equivalent if the only other cohomology is of Hodge-Tate type.

Lai and I have found such diagrams for more general Fano fourfolds.

A stronger conjecture

Conjecture (Kuznetsov* Conjecture)

A cubic fourfold is rational if and only if it admits an associated K3 surface.

Kuznetsov originally expressed this in derived category language. Addington-Thomas – taken off-the-shelf – applies to dense open subsets of the appropriate $\mathcal{C}_d$. The recent theorem by Kontsevich and Tschinkel on specialization of rationality implies the statement above.

Question

Is the derived category condition in Kuznetsov's conjecture stable under smooth specialization?

A proof was recently announced by Arend Bayer.

Cubic fourfolds and twisted K3 surfaces

Definition

A polarized K3 surface (S, f) is twisted associated *with a cubic fourfold X if we have inclusions of Hodge structures*

$$H^2(S, \mathbb{Z})_\circ(-1) \xhookleftarrow{\iota} \Lambda \xhookrightarrow{j} H^4(X, \mathbb{Z})$$

where j is saturated and ι has cyclic cokernel.

Λ is characterized as the kernel of a homomorphism

$$\alpha : H^2(S, \mathbb{Z})_\circ \to \mathbb{Q}/\mathbb{Z},$$

the twisting data when $\mathrm{Pic}(S) = \mathbb{Z}f$. Huybrechts has shown a general $[X] \in \mathcal{C}_d$ admits a twisted associated K3 if and only if

$$d/2 = \prod_i p_i^{n_i}$$

where n_i is even when $p_i \equiv 2 \pmod 3$.

Examples motivated by the classification

Tabulation of discriminants

d	**8**	12	14	**18**	20	24	**26**	30	32	36	**38**	42
K3	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$+$
twisted K3	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$-$	$+$	$-$	$+$	$+$
order(α)	2		1	3		2	1		4		1	1

d	44	48	50	54	56	60	62	66	68	72	74	78
K3	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$+$
twisted K3	$-$	$-$	$+$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$	$+$
order(α)			5	3	2		1			2	1	1

Twisted structures and rationality

The first result goes back to the 1990's:

Theorem

Each $X \in \mathcal{C}_8$, containing a plane P, yields a twisted K3 surface (S, f, α) of degree two and order two. X is rational when α vanishes in $\mathrm{Br}(S)$.

Idea: projecting from P gives a quadric surface bundle $\mathrm{Bl}_P(X) \to \mathbb{P}^2$ which is rational when the Brauer class vanishes. The second is more recent

Theorem (AHTV 2016)

$X \in \mathcal{C}_{18}$ yields a twisted K3 surface (S, f, α) of degree two and order three. X is rational when α vanishes in $\mathrm{Br}(S)$.

Idea: Fiber in sextic del Pezzo surfaces.

Twisting questions

Challenge: Give more examples along these lines, especially for higher torsion orders.
The case of $d = 50$ looks quite intriguing. How can we make sense of five torsion?
The fibrations in surfaces we use do not obviously generalize:

> *Does there exist a class of geometrically rational surfaces Σ/K (say, $K = \mathbb{C}(\mathbb{P}^2)$) whose rationality over K is controlled by an element $\alpha \in \mathrm{Br}(L)$ with order prime to 6, where L/K is a finite extension depending on Σ?*

Associated K3 surfaces and rationality

Here are new and surprising results:

Theorem (Russo-Staglianò 2017)

$X \in \mathcal{C}_{26}$, containing a septic scroll with three transverse double points, is rational.
$X \in \mathcal{C}_{38}$, containing a degree-ten surface isomorphic to $\mathbb{P}^2$ blown up in ten points, is rational.

These are the first new divisorial examples predicted by Kuznetsov, which looks much more plausible than a year ago.
The construction uses families of conics 5-secant to a prescribed surface; the family B happens to be rational. Each of these meets a cubic fourfold in six points, so the residual point of intersection gives $B \overset{\sim}{\dashrightarrow} X$.

Parametrization questions

Challenge: Describe the parametrization $\rho : \mathbb{P}^4 \to X$ in the Russo-Staglianò examples.
Does it blow up an associated K3 surface?
Give explicit linear series on X inducing ρ^{-1}.

Question

Can the rationality construction be extended to $d = 42$? (Lai)

Are there rationality constructions associated with degree e rational curves $(3e - 1)$-secant to a suitable surface? (Yes for $e = 1, 2$!)