Exponential frames and syndetic Riesz sequences

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Abstract

Employing the solution to the Kadison-Singer problem, we deduce that every subset $\mathcal{S}$ of the torus of positive Lebesgue measure admits a Riesz sequence of exponentials $\{e^{i\lambda x}\}_{\lambda\in\Lambda}$ such that $\Lambda \subset \mathbb{Z}$ is a set with gaps between consecutive elements bounded by $\dfrac{C}{|\mathcal{S}|}$. This talk is based on a joint work with Itay Londner (Tel Aviv University).

$\mathcal{H}$ separable Hilbert space, I a countable set.

Definition

$\{\varphi_i\}_{i\in I} \subset \mathcal{H}$ is a *frame* with bounds $0 < A \le B < \infty$ if

$$A\,\|f\|^2 \le \sum_{i\in I} |\langle f, \varphi_i\rangle|^2 \le B\,\|f\|^2$$

for all vectors $f \in \mathcal{H}$. $\{\varphi_i\}_{i\in I}$ is a *Bessel sequence* if $A = 0$.

Definition

$\{\varphi_i\}_{i\in I} \subset \mathcal{H}$ a *Riesz sequence* in $\mathcal{H}$ with bounds $0 < A \le B < \infty$ if

$$A \sum_{i\in I} |a_i|^2 \le \left\| \sum_{i\in I} a_i \varphi_i \right\|_{\mathcal{H}}^2 \le B \sum_{i\in I} |a_i|^2$$

for every finite sequence of scalars $\{a_i\}_{i\in I}$.

Exponential systems

Definition

$\mathcal{S} \subset \mathbb{R}$ set of finite positive Lebesgue measure, $\Lambda \subset \mathbb{R}$ a countable set. Define exponential system $E(\Lambda) = \{e^{i\lambda x}\}_{\lambda \in \Lambda}$ in $L^2(\mathcal{S})$.

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S bounded and Λ separated set $\inf_{\lambda \neq \mu} |\lambda - \mu| > 0$
$\implies E(\Lambda)$ is Bessel.

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S bounded and Λ separated set $\inf_{\lambda \neq \mu} |\lambda - \mu| > 0$
$\implies E(\Lambda)$ is Bessel.
$\mathcal{S} \subset \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ set of positive Lebesgue measure
$\implies E(\mathbb{Z})$ is a Parseval frame in $L^2(\mathcal{S})$.

Classical results

Theorem (Kahane (1957))

Let $I \subset \mathbb{R}$ be an interval. If the upper density

$$D^{+}(\Lambda) := \lim_{r \to \infty} \sup_{a \in \mathbb{R}} \frac{\#(\Lambda \cap (a, a+r))}{r} < \frac{|I|}{2\pi},$$

then $E(\Lambda)$ is a Riesz sequence in $L^2(I)$. On the other hand if $D^{+}(\Lambda) > \frac{|I|}{2\pi}$ then $E(\Lambda)$ is not a Riesz sequence in $L^2(I)$.

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Theorem (Landau (1967))

Let $\mathcal{S}$ be a measurable set. If $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$ then $D^{+}(\Lambda) \leq \frac{|\mathcal{S}|}{2\pi}$.

Question

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This question may be considered under various notions of density. The first result on this subject is

Theorem (Bourgain-Tzafriri (1987))

Given $\mathcal{S} \subset \mathbb{T}$ of positive measure, there exists a set $\Lambda \subset \mathbb{Z}$ with positive asymptotic density

$$dens(\Lambda) = \lim_{r\to\infty} \frac{\#(\Lambda \cap (-r, r))}{2r} > c\,|\mathcal{S}|$$

and such that $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$.
Here c is an absolute constant, independent of $\mathcal{S}$.

Hence, every set $\mathcal{S}$ admits a Riesz sequence Λ with positive upper density, proportional to the measure of $\mathcal{S}$.

Definition

A subset $\Lambda = \{\ldots < \lambda_0 < \lambda_1 < \lambda_2 < \ldots\} \subset \mathbb{Z}$ is *syndetic* if the consecutive gaps in Λ are bounded

$$\gamma(\Lambda) := \sup_{n \in \mathbb{Z}} (\lambda_{n+1} - \lambda_n) < \infty.$$

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Theorem (Lawton (2010) and Paulsen (2011))

Given a set $\mathcal{S} \subset \mathbb{T}$ of positive measure, TFAE:

(i) *There exists $r \in \mathbb{N}$ and a partition $\mathbb{Z} = \bigcup_{j=1}^{r} \Lambda_j$ such that $E(\Lambda_j)$ is a Riesz sequences in $L^2(\mathcal{S})$ for all $j = 1, \ldots, r$.*

(ii) *There exists $d \in \mathbb{N}$ and a syndetic set $\Lambda \subseteq \mathbb{Z}$ with $\gamma(\Lambda) = d$ such that $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$.*

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Remark

(ii) $\implies$ (i) can take $r \leq d$ by considering translates of Λ.
(i) $\implies$ (ii) no upper bound on d in terms of r.

Remark

Statement (i) is known as the Feichtinger conjecture for exponentials. The Feichtinger conjecture in its general form states that every bounded frame can be decomposed into finitely many Riesz sequences. It has been proved to the Kadison-Singer problem by Casazza-Christensen-Lindner-Vershynin (2005) and Casazza-Tremain (2006). The latter has been solved by Marcus, Spielman and Srivastava (2013).

Solution of Kadison-Singer Problem

Theorem (Marcus-Spielman-Srivastava (2013))

If $\varepsilon > 0$ and $v_1, \ldots, v_m$ are independent random vectors in $\mathbb{C}^d$ with finite support such that $\sum_{i=1}^{m} \mathbb{E}\left[v_i v_i^*\right] \leq \mathbf{I}_d$ and $\mathbb{E}\left[\|v_i\|^2\right] \leq \varepsilon$ for all i, then

$$\mathbb{P}\left(\left\|\sum_{i=1}^{m} v_i v_i^*\right\| \leq \left(1+\sqrt{\varepsilon}\right)^2\right) > 0.$$

Improvement for support of size 2

Theorem (B.-Casazza-Marcus-Speegle (2016))

If $\varepsilon \in \left(0, \frac{1}{2}\right)$ *and* $v_1, \ldots, v_m$ *are independent random vectors in* $\mathbb{C}^d$ *with support of size 2 such that* $\sum_{i=1}^{m} \mathbb{E}\left[v_i v_i^*\right] \leq \mathbf{I}_d$ *and* $\mathbb{E}\left[\|v_i\|^2\right] \leq \varepsilon$ *for all* i*, then*

$$\mathbb{P}\left(\left\|\sum_{i=1}^{m} v_i v_i^*\right\| \leq 1 + 2\sqrt{\varepsilon\left(1-\varepsilon\right)}\right) > 0.$$

Feichtinger conjecture

Theorem (B.-Casazza-Marcus-Speegle (2016))

Let $\varepsilon > 0$ and suppose that $\{u_i\}_{i \in I}$ is a Bessel sequence in $\mathcal{H}$ with bound 1 that consists of vectors of norms $\|u_i\|^2 \geq \varepsilon$. Then there exists a universal constant $C > 0$, such that I can be partitioned into $r \leq \frac{C}{\varepsilon}$ subsets $I_1, \ldots, I_r$ such that every subfamily $\{u_i\}_{i \in I_j}$, $j = 1, \ldots, r$ is a Riesz sequence in $\mathcal{H}$. Moreover, if $\varepsilon > 3/4$, then $r = 2$ works.

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Corollary

There exists a universal constant $C > 0$ such that for any subset $\mathcal{S} \subset \mathbb{T}$ with positive measure, the exponential system $E(\mathbb{Z})$ can be decomposed as a union of $r \leq \frac{C}{|\mathcal{S}|}$ Riesz sequences $E(\Lambda_j)$ in $L^2(\mathcal{S})$ for $j = 1, \dots, r$. Moreover, if $|\mathcal{S}| > 3/4$, then $r = 2$ works.

Lawton's Theorem and the solution of Kadison-Singer problem (Feichtinger conjecture) yield syndetic Riesz sequences of exponentials in $L^2(\mathcal{S})$.

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Question (Olevskii)

What is the bound on a gap $\gamma(\Lambda)$ *for syndetic* $\Lambda \subset \mathbb{Z}$ *such that* $E(\Lambda)$ *is a Riesz sequence in* $L^2(\mathcal{S})$*?*

Main result

Theorem (B.-Londner (2018))

Let $\varepsilon > 0$ *and suppose that* $\{u_i\}_{i \in I}$ *is a Bessel sequence in* $\mathcal{H}$ *with bound* 1 *and*

$$\|u_i\|^2 \geq \varepsilon \qquad \forall i \in I.$$

Then there exists a universal constant $C > 0$ *such that whenever* $\{J_k\}_k$ *is a collection of disjoint subsets of* I *with* $\# J_k \geq r = \lceil \frac{C}{\varepsilon} \rceil$, *for all* k. *There exists a selector, i.e. a subset* $J \subset \bigcup_k J_k$ *satisfying*

$$\#(J \cap J_k) = 1 \qquad \forall k$$

and such that $\{u_i\}_{i \in J}$ *is a Riesz sequence in* $\mathcal{H}$. *Moreover, if* $\varepsilon > \frac{3}{4}$ *then the same conclusion holds with* $r = 2$.

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and such that $\{u_i\}_{i\in J}$ is a Riesz sequence in $\mathcal{H}$. Moreover, if $\varepsilon > \frac{3}{4}$ then the same conclusion holds with $r = 2$.

Applying this to the exponential system $\{\mathbf{1}_\mathcal{S} e^{2\pi int}\}_{n\in\mathbb{Z}}$ with $J_k = [kr, (k+1)r) \cap \mathbb{Z}$, $k \in \mathbb{Z}$, yields:

Corollary

There exists a universal constant $C > 0$ *such that for any subset* $\mathcal{S} \subset \mathbb{T}$ *with positive measure, there exists a syndetic set* $\Lambda \subset \mathbb{Z}$ *with gaps* $\gamma(\Lambda) \leq C |\mathcal{S}|^{-1}$ *so that* $E(\Lambda)$ *is a Riesz sequence in* $L^2(\mathcal{S})$*. Moreover, if* $|\mathcal{S}| > \frac{3}{4}$ *then such* Λ *exists with* $\gamma(\Lambda) \leq 3$*.*

Corollary

There exists a universal constant $C > 0$ such that for any subset $\mathcal{S} \subset \mathbb{T}^d$ of positive measure, any d-dimensional rectangle $\mathcal{R} \subset \mathbb{Z}^d$ with $\# |\mathcal{R}| > C |\mathcal{S}|^{-1}$, and any parition $\mathbb{Z}^d = \bigcup \mathcal{R}_k$ into disjoint union of translated copies of $\mathcal{R}$, there exists a set $\Lambda \subset \mathbb{Z}^d$ such that

$$\#|\Lambda \cap \mathcal{R}_k| = 1 \qquad \forall k$$

and $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$. If $\mathcal{R}$ is a cube, then

$$\sup_{\lambda \in \Lambda} \inf_{\mu \in \Lambda \setminus \{\lambda\}} |\lambda - \mu| \le C\sqrt{d}\,|\mathcal{S}|^{-\frac{1}{d}}.$$

Higher dimensional syndetic sets 2

Partitioning the lattice $\mathbb{Z}^d$ into thin and long rectangles in a checkerboard way yields

Corollary

There exists a universal constant $C > 0$ *such that for any subset* $\mathcal{S} \subset \mathbb{T}^d$ *of positive measure, there exists a set* $\Lambda \subset \mathbb{Z}^d$ *so that* $E(\Lambda)$ *is a Riesz sequence in* $L^2(\mathcal{S})$ *and every one dimensional section of* Λ*, i.e. every set of the form*

$$\Lambda(k_1, \ldots, k_{d-1}) = \{k \in \mathbb{Z} : (k_1, \ldots, k_{d-1}, k) \in \Lambda\}$$

is syndetic for any $(k_1, \ldots, k_{d-1}) \in \mathbb{Z}^{d-1}$*, and*

$$\gamma(\Lambda(k_1, \ldots, k_{d-1})) \leq Cd\,|\mathcal{S}|^{-1}.$$

Basic selector theorem

Theorem

Let $r, M \in \mathbb{N}$ and $\delta > 0$. Suppose that $\{u_i\}_{i=1}^M \subset \mathcal{H}$ is a Bessel sequence with bound 1 and $\|u_i\|^2 \leq \delta$ for all i. Then for every collection of disjoint subsets $J_1, \ldots, J_n \subset [M]$ with $\# J_k \geq r$ for all k, there exists a subset $J \subset [M]$ such that $\#(J \cap J_k) = 1$ for all $k \in [n]$ and the system of vectors $\{u_i\}_{i \in J}$ is a Bessel sequence with bound

$$\left(\frac{1}{\sqrt{r}} + \sqrt{\delta} \right)^2 .$$

Proof.

WLOG $\#J_k = r$. Define independent random vectors v_k: for $k = 1, \ldots, n$ the vector v_k takes values $\sqrt{r}u_i$ for any $i \in J_k$ with probability $\frac{1}{r}$. Then,

$$\sum_{k=1}^{n} \mathbb{E}\left(v_k v_k^*\right) \leq \mathbf{I}_{\mathcal{H}} \qquad \text{and} \qquad \mathbb{E}\left\|v_k\right\|^2 \leq r\delta \qquad \forall k.$$

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By Theorem of Marcus-Spielman-Srivastava

$$\mathbb{P}\left(\left\|\sum_{k=1}^{n} v_k v_k^*\right\| \le \left(1+\sqrt{r\delta}\right)^2\right) > 0.$$

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$$\mathbb{P}\left(\left\| \sum_{k=1}^{n} v_k v_k^* \right\| \le \left(1 + \sqrt{r\delta}\right)^2 \right) > 0.$$

which implies the existence of a set $J \subset [M]$ such that

$$\left\| \sum_{i \in J} u_i u_i^* \right\| \le \left(\frac{1}{\sqrt{r}} + \sqrt{\delta} \right)^2 .$$

Selector theorem for short vectors

Theorem

Let $M \in \mathbb{N}$ and $\delta_0 \in \left(0, \frac{1}{4}\right)$. Suppose that $\{u_i\}_{i=1}^M \subset \mathcal{H}$ is a Bessel sequence with Bessel bound 1 and $\|u_i\|^2 \leq \delta_0$ for all i. Then for every collection of disjoint subsets $J_1, \ldots, J_n \subset [M]$ with $\# J_k = 2$ for all k, there exists a subset $J \subset [M]$ such that $\# (J \cap J_k) = 1$ for all $k \in [n]$ and the system of vectors $\{u_i\}_{i \in J}$ is a Bessel sequence with bound $1 - \varepsilon_0$, where $\varepsilon_0 = \frac{1}{2} - \sqrt{2\delta_0 (1 - 2\delta_0)}$.

Naimark's complements

Lemma (B.-Casazza-Marcus-Speegle (2016))

Let $P : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto a closed subspace $H \subset \mathcal{H}$, and let $\{e_i\}_{i \in I}$ be an orthogonal basis for $\mathcal{H}$. Then for any subset $J \subset I$ and $\delta > 0$ the following are equivalent:

1. $\{Pe_i\}_{i \in J}$ *is a Bessel sequence with bound* $1 - \delta$.
2. $\{(\boldsymbol{I}_{\mathcal{H}} - P)\, e_i\}_{i \in J}$ *is a Riesz sequence with lower bound* δ.

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Corollary

Let $M \in \mathbb{N}$ and $\delta_0 \in \left(0, \frac{1}{4}\right)$. Suppose that $\{u_i\}_{i=1}^{M} \subset \mathcal{H}$ is a Bessel sequence with Bessel bound B and $\|u_i\|^2 \geq B\,(1 - \delta_0)$ for all i. Then for every collection of disjoint subsets $J_1, \ldots, J_n \subset [M]$ with $\# J_k = 2$ for all k, there exists a subset $J \subset [M]$ such that $\#\,(J \cap J_k) = 1$ for all $k \in [n]$ and the system of vectors $\{u_i\}_{i \in J}$ is a Riesz sequence with lower Riesz bound $B\varepsilon_0$.

Combine two selectors theorems with

Lemma

Let $\mathcal{H}$ be an infinite dimensional Hilbert space, $M \in \mathbb{N}$ and $\delta \in (0,1)$. Suppose $\{u_i\}_{i=1}^M \subset \mathcal{H}$ is a Bessel sequence with Bessel bound 1 and $\|u_i\|^2 \geq \delta$ for all i. Then for every large enough $K \in \mathbb{N}$, there exist vectors $\varphi_1, \ldots, \varphi_K \in \mathcal{H}$ with $\|\varphi_i\|^2 \geq \delta$ for all i such that $\{u_i\}_{i=1}^M \cup \{\varphi_i\}_{i=1}^K$ is a Parseval frame for its linear span.

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Theorem (Finite version of main result)

Let $\varepsilon > 0$ and $M \in \mathbb{N}$. Suppose that $\{u_i\}_{i=1}^M \subset \mathcal{H}$ is a Bessel sequence with Bessel bound 1 and $\|u_i\|^2 \geq \varepsilon$ for all i. Then there exists $r = O\left(\frac{1}{\varepsilon}\right)$, independent of M, such that for every collection of disjoint subsets $J_1, \dots, J_n \subset [M]$ with $\# J_k \geq r$ for all k, there exists a subset $J \subset [M]$ such that $\#(J \cap J_k) = 1$ for all $k \in [n]$ and the system of vectors $\{u_i\}_{i \in J}$ is a Riesz sequence with lower Riesz bound $\varepsilon\varepsilon_0$. Moreover, if $\varepsilon > \frac{3}{4}$ then the same conclusion holds with $r = 2$.

Diagonal argument with the pigeonhole principle

Lemma

Let $\{J_k\}_k$ be a collection of disjoint subsets of I. Assume for every $n \in \mathbb{N}$ we have a subset $I_n \subset \bigcup_{k=1}^n J_k$ such that

$$\#(I_n \cap J_k) = 1 \quad \text{for } k = 1, \ldots, n$$

Then, there exists a subset $I_\infty \subset I$ and an increasing sequence $\{n_j\}$ such that

$$I_{n_j} \cap \left(\bigcup_{k=1}^{j} J_k \right) = I_\infty \cap \left(\bigcup_{k=1}^{j} J_k \right)$$

In particular, we have

$$\#(I_\infty \cap J_k) = 1 \qquad \forall k.$$

This yields infinite dimensional version of main result.

Syndetic sets and almost tight Riesz bounds

Theorem (R_ε conjecture of Casazza-Tremain)

Let $\{u_i\}_{i\in I}$ be a unit norm Bessel sequence in $\mathcal{H}$ with bound B. Then there exists a universal constant $C>0$ such that for any $\varepsilon>0$ and any collection of disjoint subsets of I, $\{J_k\}_k$ satisfying $\# J_k \ge r = \lceil C\frac{B}{\varepsilon^4}\rceil$, for all k. There exists a selector $J \subset \bigcup_k J_k$ satisfying

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and such that $\{u_i\}_{i\in J}$ is a Riesz sequence in $\mathcal{H}$ with bounds $1\pm\varepsilon$.

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Remark

A multi-paving result of Ravichandran-Srivastava (2017) suggest that $r = O(\frac{B}{\varepsilon^2})$ should work. Hence, this would yield syndetic Riesz sequences of exponentials in $L^2(\mathcal{S})$ with Riesz bounds $|\mathcal{S}|(1 \pm \varepsilon)$ and gaps $O(\frac{1}{|\mathcal{S}|\varepsilon^2})$ instead of $O(\frac{1}{|\mathcal{S}|\varepsilon^4})$.

Olevskii and Ulanovskii asked whether there exists a set Λ such that $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$ for all sets $\mathcal{S} \subset \mathbb{T}$ with large measure. They proved that the answer, in general, is negative.

Theorem (Olevskii-Ulanovskii (2008))

Let $d \in (0,1)$. Then for every $\varepsilon \in (0,1)$ and $\Lambda \subset \mathbb{R}$ with $D(\Lambda) = d$, there exists a set $\mathcal{S} \subset \mathbb{T}$ with $\frac{|\mathcal{S}|}{2\pi} > 1 - \varepsilon$ such that $E(\Lambda)$ is not a Riesz sequence in $L^2(\mathcal{S})$.

Universal Riesz sequences

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On the other hand, restricting to open sets they showed

Theorem (Olevskii-Ulanovskii (2008))

For every $d \in (0,1)$, there is a universal Riesz sequence, i.e. a set $\Lambda \subset \mathbb{R}$ with $D(\Lambda) = d$ such that $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$ for every open set $\mathcal{S} \subset \mathbb{T}$ with $\frac{|\mathcal{S}|}{2\pi} > d$.

Definition (Meyer (1972))

Let α be an irrational number, and $I = [a, b) \subset [0, 1]$. The *(simple) quasicrystal* corresponding to α and I is

$$\Lambda(\alpha, I) = \{n \in \mathbb{Z} \mid \{\alpha n\} \in I\}$$

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Since sequence $\{\alpha n\}$ is equidistributed in $[0, 1]$, the corresponding simple quasicrystal has uniform density $D(\Lambda(\alpha, I)) = |I|$.

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$$\Lambda(\alpha, I) = \{n \in \mathbb{Z} \mid \{\alpha n\} \in I\}$$

where $\{x\}$ is the fractional part of the real number x.

Since sequence $\{\alpha n\}$ is equidistributed in $[0, 1]$, the corresponding simple quasicrystal has uniform density $D(\Lambda(\alpha, I)) = |I|$.

Theorem (Matei-Meyer (2009))

The system $E(\Lambda(\alpha, I))$ is a universal Riesz sequence, i.e., $E(\Lambda(\alpha, I))$ is a Riesz sequence in $L^2(\mathcal{S})$ for every open set $\mathcal{S} \subset \mathbb{T}$ with $\frac{|\mathcal{S}|}{2\pi} > |I|$.

Quasicrystal methods yield a deterministic construction of Riesz syndetic sets Λ. Moreover, for sets $\mathcal{S} \subset \mathbb{T}$ of almost full measure $|\mathcal{S}| \to 1$ we can remove from $\mathbb{Z}$ a syndetic set with gaps sizes at least $C/(1-|\mathcal{S}|)$.

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Theorem (B.-Londner (2018))

There exists a universal constant $C > 0$ such that for any open subset $\mathcal{S} \subset \mathbb{T}$, one can construct a syndetic set $\Lambda \subset \mathbb{Z}$ with gaps between consecutive elements taking exactly two values $\{1, d\}$, where $d = d(\mathcal{S}) \le \frac{C}{|\mathcal{S}|}$, and so that $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$. Moreover, Λ can be chosen so that $\Lambda^c = \mathbb{Z} \backslash \Lambda$ satisfies

$$\inf_{\lambda,\mu \in \Lambda^c, \lambda \neq \mu} |\lambda - \mu| \ge \frac{C}{|S^c|}.$$

Open problems on syndetic Riesz sequences

Question (1)

Olevskii-Ulanovskii Theorem that $E\left(\Lambda\left(\alpha, I\right)\right)$ is not a Riesz sequence in $L^2\left(\mathcal{S}\right)$ for some non-open set $\mathcal{S}$. Does there exist $\mathcal{S}$ with empty interior for which $E\left(\Lambda\left(\alpha, I\right)\right)$ is a Riesz sequence? Otherwise prove that such set does not exist.

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Does an arbitrary measurable set $\mathcal{S}$ of positive measure admit an exponential Riesz sequence $E(\Lambda)$, $\Lambda \subset \mathbb{Z}$, such that

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Question (Olevskii)

Does an arbitrary measurable set $\mathcal{S}$ of positive measure admit an exponential Riesz **basis** *$E(\Lambda)$ for some $\Lambda \subset \mathbb{R}$?*

THANK YOU FOR ATTENTION