Frames generated from exponential of operators

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Outline

1. Statement of the problem

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2. Frames induced by powers of an operator

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3. Semi-continuous frames induced by powers of an operator

Problem Statement

General Statement: Let $\mathcal{H}$ be a Hilbert space, A a bounded operator on $\mathcal{H}$, and $\mathcal{G}$ a countable subset of $\mathcal{H}$. Find conditions on A, and $\mathcal{G}$, such that the system

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Motivation: Sampling and reconstruction of functions that are evolving in time under the action of an evolution operator.

Definitions

Let $A \in B(\mathcal{H})$, $\mathcal{G} \subset \mathcal{H}$, $\tau \subset \mathbb{R}$, and $E = \{A^t g : g \in \mathcal{G}, t \in \mathcal{T}\} \subset \mathcal{H}$.

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1. E is Bessel if $\sum_{g\in\mathcal{G}} \int_{\mathcal{T}} |\langle f, A^t g\rangle|^2 d\mu(t) \leq C_2 \|f\|^2$, for all $f \in \mathcal{H}$

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2. E is a (semi-continuous) frame for $\mathcal{H}$ if there exists $C_1, C_2 > 0$ such that

$$C_1 \|f\|^2 \leq \sum_{g \in \mathcal{G}} \int_{\mathcal{T}} |\langle f, A^t g\rangle|^2 d\mu(t) \leq C_2 \|f\|^2.$$

Theorem. [A., Armenak Petrosian– 2017] *If for* $A \in B(\mathcal{H})$ *there exists* $\mathcal{G} \subset \mathcal{H}$ *such that* $\{A^n g\}_{g \in \mathcal{G}, n \in \mathbb{N}}$ *(*$L = \infty$*) is a frame for* $\mathcal{H}$*, then for every* $f \in \mathcal{H}$*,* $(A^*)^n f \to 0$ *as* $n \to \infty$*.*

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Example: $\mathcal{H} = \ell^2(\mathbb{N})$, $Ax = (0, x_1, x_2, \dots)$, $\mathcal{G} = \{e_1 = (1, 0, \dots)\}$. Then, $A^n e_1 = A^n(1, 0, \dots) = e_n$. Thus, $\{A^n g\}_{g \in \mathcal{G}, n \in \mathbb{N}}$ is a frame for $\mathcal{H}$ (orthonormal basis). $(A^*)^n x = (x_n, x_{n+1}, \dots) \to 0$.

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Corollary. [A., Armenak Petrosian– 2017] *If* $A \in B(\mathcal{H})$ *is unitary, then no* $\mathcal{G}$ *is such that* $\{A^n g\}_{g \in \mathcal{G}, n \in \mathbb{N}}$ *is a frame for* $\mathcal{H}$*.*

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More recent: Christensen, Hasannasabjaldehbakhani and Philipp– ICCHA7–2018): Also $A^n f \to 0$ as $n \to \infty$.

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Characterization (Cabrelli, Molter, Paternostro, Philipp – 2018)

(1) $(A^*)^n f \to 0$, $f \in \mathcal{H}$; (2) $\mathcal{G}$ is Bessel; and (3) There exists an invertible S such that $ASA^* = S - G$ where $Gf = \sum_g \langle f, g \rangle g$. Then $\{A^n g\}_{g \in \mathcal{G}, n \in \mathbb{N}}$ is a frame for $\mathcal{H}$.

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S is necessarily the frame oparator $Sf = \sum_{g \in G, n \geq 0} \langle f, A^n g \rangle A^n g$

Finite set $\mathcal{G}$

Theorem. [A., Armenak Petrosian– 2017] *If* $\dim \mathcal{H} = \infty$, $|\mathcal{G}| < \infty$, *and* $\{A^n g\}_{g\in\mathcal{G}, n\in\mathbb{N}}$ *is a frame for* $\mathcal{H}$, *then* $\|A\| \geq 1$.

Theorem. [A., Cabrelli, Cakmak, Molter, Pertrosyan–JFA 17] *Let* $A \in B(\mathcal{H})$ *be normal,* $\dim(\mathcal{H}) = \infty$. *Then,* $\{A^n g\}_{n \geq 0}$ *is a frame for* $\mathcal{H}$ *if and only if*

1. $A = \sum_{j \in \mathbb{N}} \lambda_j P_j$, *where* P_j *are rank one ortho-projections.*

2. $|\lambda_k| < 1$ *for all* k, *and* $|\lambda_k| \to 1$ *and* $\{\lambda_k\}$ *satisfies Carleson condition* $\inf_n \prod_{k \neq n} \frac{|\lambda_n - \lambda_k|}{|1 - \bar{\lambda}_n \lambda_k|} \geq \delta$ *for some* $\delta > 0$.

3. $0 < C_1 \leq \frac{\|P_j g\|}{\sqrt{1 - |\lambda_j|^2}} \leq C_2 < \infty$ *for some constants* C_1, C_2.

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Conjecture: If A is a normal operator on $\mathcal{H}$, then the system $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathcal{G},n\geq 0}$ is not a frame for $\mathcal{H}$.

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(conjecture is true for A self-adjoint (A., Cabrelli, Molter, Tang– 2017).

Semi-continuous frames

Theorem. [A., Petrosyan, and Huang–2017] *Let* $A \in \mathcal{B}(\mathcal{H})$ *be normal, and* $\mathcal{G}$ *Bessel. If* $\{A^t g\}_{g\in\mathcal{G}, t\in[0,L]}$ *is a semi-continuous frame for* $\mathcal{H}$*, then* $\exists\ \delta > 0$ *s.t. for any finite set* $T = \{t_i : i = 1, \ldots, n\}$ *with* $0 = t_1 < t_2 < \ldots < t_n < t_{n+1} = L$ *and* $|t_{i+1} - t_i| < \delta$*, the system* $\{A^t g\}_{g\in\mathcal{G}, t\in T}$ *is a frame for* $\mathcal{H}$*.*

If, in addition, A *is invertible, then* $\{A^t g\}_{g\in\mathcal{G}, t\in[0,L]}$ *is a semi-continuous frame for* $\mathcal{H}$ *if and only if there exists a finite set* $T = \{t_i : i = 1, \ldots, n\}$ *and* $0 = t_1 < t_2 < \ldots < t_n < L$*, such that* $\{A^t g\}_{g\in\mathcal{G}, t\in T}$ *is a frame for* $\mathcal{H}$*.*

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Thus, if $\dim(\mathcal{H}) = \infty$, then $\mathcal{G}$ must be infinite.

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Discretization of continuous frames (Freeman and Speegle).

Theorem. [A., Petrosyan, and Huang–2017] *Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint invertible operator and $\mathcal{G}$ be a countable set in $\mathcal{H}$. Then $\{A^t g\}_{g\in\mathcal{G},t\in[0,1]}$ is a frame in $\mathcal{H}$ iff $\{A^t g\}_{g\in\mathcal{G},t\in[0,L]}$ is a frame in $\mathcal{H}$ for all finite positive L.*

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Conjecture: Same is true for normal operators.

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Thank you