Fuglede's spectral set conjecture on cyclic groups

Romanos Diogenes Malikiosis

TU Berlin

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& work in progress

Fourier Analysis on domains $\Omega \subseteq \mathbb{R}^n$

Question

On which measurable domains $\Omega \subseteq \mathbb{R}^n$ with $\mu(\Omega) > 0$ can we do Fourier analysis, that is, there is an orthonormal basis of exponential functions $\left\{ \frac{1}{\mu(\Omega)} e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda \right\}$ in $L^2(\Omega)$, where $\Lambda \subseteq \mathbb{R}^n$ discrete?

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Conjecture (Fuglede, 1974)

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Let $e_\lambda(x) = e^{2\pi i\lambda\cdot x}$. Wlog, $\mu(\Omega) = 1$. Inner product and norm on $L^2(\Omega)$:

$$\langle f, g\rangle_\Omega = \int_\Omega f\bar{g}, \quad \|f\|_\Omega^2 = \int_\Omega |f|^2.$$

It holds $\langle e_\lambda, e_\mu\rangle_\Omega = \widehat{\mathbf{1}_\Omega}(\mu - \lambda)$.

Lemma

Λ is a spectrum of Ω if and only if

$$\widehat{\mathbf{1}_\Omega}(\lambda - \mu) = 0, \ \forall \lambda \neq \mu, \lambda, \mu \in \Lambda$$

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Special cases

Theorem (Fuglede, '74)

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set of measure 1 and $\Lambda \subseteq \mathbb{R}^n$ be a lattice with density 1. Then $\Omega \oplus \Lambda = \mathbb{R}^n$ if and only if $\Lambda^\star$ is a spectrum of Ω.

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According to the theorems of Venkov ('54) and McMullen ('80), the above do not tile $\mathbb{R}^n$.

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Passage to finite groups

Definition

Let G be an Abelian group. We write **(S-T**(G)**)** if every bounded spectral subset of G is also a tile, and **(T-S**(G)**)** if every bounded tile of G is spectral.

Theorem (Dutkay, Lai, '14)

The following hold:

$$\textbf{(T-S}(\mathbb{Z}_n)\textbf{)} \forall n \in \mathbb{N} \Leftrightarrow \textbf{(T-S}(\mathbb{Z})\textbf{)} \Leftrightarrow \textbf{(T-S}(\mathbb{R})\textbf{)}$$

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Definition (Coven-Meyerowitz, '98)

Let $A \subseteq \mathbb{Z}_N$. The mask polynomial A is given by

$$\sum_{a \in A} X^a \in \mathbb{Z}[X]/(X^N - 1).$$

It holds

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The properties (T1) and (T2)

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$$S_A = \{d \mid N : d \text{ prime power}, A(\zeta_d) = 0\}.$$

We define the following properties:

(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$

(T2) Let $s_1, s_2, \ldots, s_k \in S_A$ be powers of different primes. Then $\Phi_s(X) \mid A(X)$, where $s = s_1 \cdots s_k$.

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When N is a prime power, (T2) holds vacuously. If $N = p^n q^m$, then (T2) is simply

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Example

Let $A \subseteq \mathbb{Z}_N$, $N = p^4 q^4 r^3$, such that

$$A(\zeta_p) = A(\zeta_{p^3}) = A(\zeta_{q^2}) = A(\zeta_{r^3}) = 0,$$

and $A(X)$ has no other root of order a power of p, q, or r. Then, (T1) is equivalent to $|A| = p^2 qr$, and (T2) is equivalent to

$$A(\zeta_{pq^2}) = A(\zeta_{p^3 q^2}) = A(\zeta_{pr^3}) = A(\zeta_{p^3 r^3}) = A(\zeta_{q^2 r^3}) = \\ = A(\zeta_{pq^2 r^3}) = A(\zeta_{p^3 q^2 r^3}) = 0.$$

Tiling, spectrality, and (T1), (T2)

The following are consequences of the works of Coven-Meyerowitz ('98) and Łaba ('02); also Kolountzakis-Matolcsi ('07).

Theorem

If $A \subseteq \mathbb{Z}_N$ satisfies (T1) and (T2), then it tiles $\mathbb{Z}_N$. If A tiles $\mathbb{Z}_N$, then it satisfies (T1); if in addition $N = p^n q^m$, then A satisfies (T2) as well.

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If $A \subseteq \mathbb{Z}_N$ satisfies (T1) and (T2), then it is spectral. If $N = p^n$ and A is spectral, then it satisfies (T1).

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(T-S($\mathbb{Z}_N$)), N square-free

Let $A \oplus T = \mathbb{Z}_N$, with $|A| = m$. Then, also $A \oplus mT = \mathbb{Z}_N$, due to

$$(A - A) \cap (T - T) = \{0\}.$$

The mask polynomial of mT is $T(X^m) \bmod (X^N - 1)$, so if $p_1, \ldots, p_k \mid m$, we have

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Primitive subsets of $\mathbb{Z}_N$

Definition

A subset $A \subseteq G$ is called *primitive* if it is not contained in a proper coset of G.

Lemma

Let $G = \mathbb{Z}_N$ with $N = p^n q^m$, and $A \subseteq \mathbb{Z}_N$ primitive. Then $(A - A) \cap \mathbb{Z}_N^\star \neq \varnothing$.

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Proof.

Let $a \in A$. Since $a - A \not\subseteq p\mathbb{Z}_N$ or $q\mathbb{Z}_N$, there are $a', a'' \in A$ such that $a - a' \notin p\mathbb{Z}_N$, $a - a'' \notin q\mathbb{Z}_N$. If either $a - a' \notin q\mathbb{Z}_N$ or $a - a'' \notin p\mathbb{Z}_N$, then we're done, so wlog $q \mid a - a'$ and $p \mid a - a''$, which yields $a'' - a' \in \mathbb{Z}_N^\star$. □

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Primitive spectral pairs, $N = p^n q^m$

Corollary

Let (A, B) be a spectral pair in $\mathbb{Z}_N$, such that both A and B are primitive. Then,

$$A(\zeta_N) = B(\zeta_N) = 0.$$

Remark

If A is not primitive, then $A \subseteq p\mathbb{Z}_N$ (say), which implies $(B - B) \cap \frac{N}{p}\mathbb{Z}_N = \{0\}$. Then, $(\overline{A}, B)$ is a spectral pair in $\mathbb{Z}_{N/p}$, where $p \cdot \overline{A} = A$. Moreover, if $\overline{A}$ satisfies (T1) and (T2) in $\mathbb{Z}_{N/p}$, then A satisfies the same properties in $\mathbb{Z}_N$.

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Vanishing sums of roots of unity

Lemma

Let $\mathrm{rad}(N) = pq$ *and* $A(X) \in \mathbb{Z}[X]$ *with nonnegative coefficients, such that* $A(\zeta_N^d) = 0$*, for some* $d \mid N$*. Then,*

$$A(X^d) \equiv P(X^d)\Phi_p(X^{N/p}) + Q(X^d)\Phi_q(X^{N/q}) \bmod (X^N - 1),$$

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If A is the disjoint union of p-cycles only, then

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We reduce to the case where both A and B are nontrivial unions of p- and q-cycles. This implies

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Proposition

Let (A, B) be a primitive spectral pair in $\mathbb{Z}_N$, $N = p^n q^m$, such that neither A nor B is a union of p-(or q-)cycles exclusively. Then, both $A(X)$ and $B(X)$ vanish at

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A special case

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If $N = p^m q^n$ *and* $A \subseteq \mathbb{Z}_N$ *is spectral satisfying*

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for some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, so it A satisfies (T2) as well. □

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Theorem

Let $A \subseteq \mathbb{Z}_N$ be spectral, with $N = p^n q^m$, $m \leq 2$. Then A tiles $\mathbb{Z}_N$.

Proof.

Wlog, A and a spectrum B are both primitive and nontrivial unions of p- and q-cycles, so using the above reductions we may assume $A(\zeta_q) = A(\zeta_{q^2}) = 0$, which by the previous Proposition yields that A tiles $\mathbb{Z}_N$. □

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The absorption-equidistribution property

Definition

We say that a subset $A \subseteq \mathbb{Z}_N$ satisfies the absorption-equidistribution property, if for every $d \mid N$ and p prime such that $pd \mid N$, either every subset $A_{j \bmod d}$ is equidistributed $\bmod pd$, that is

$$|A_{j+kd \bmod pd}| = \frac{1}{p}|A_{j \bmod d}|, \forall k \in \{0, 1, \dots, p-1\},$$

or every $A_{j \bmod d}$ is absorbed $\bmod pd$, i. e. there is $k \in \{0, 1, \dots, p-1\}$ such that

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Remark

With the above Corollary, we may further reduce to spectral (A, B), where both A, B are absorption-free.

This is used to prove:

Theorem (M)

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Thank you