Hausdorff operators in H^p spaces, $0 < p < 1$

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joint work with Akihiko Miyachi

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- In dimension one, after Kanjin, Miyachi, and Weisz, more or less final results were given in a joint paper by L-Miyachi.
- The results differ from those for L^p, $1 \le p \le \infty$, and H^1, since they involve smoothness conditions on the averaging function, which seem unusual but unavoidable.

Definitions

Given a function ϕ on the half line $(0, \infty)$, the Hausdorff operator $\mathcal{H}_\phi$ is defined by

$$(\mathcal{H}_\phi f)(x) = \int_0^\infty \frac{\phi(t)}{t} f(\frac{x}{t})\, dt, \quad x \in \mathbb{R}.$$

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- If $1 \le p \le \infty$, an application of Minkowski's inequality gives

$$\|\mathcal{H}_\phi f\|_{L^p(\mathbb{R})} \le \int_0^\infty |\phi(t)| \|\frac{1}{t} f(\frac{\cdot}{t})\|_{L^p(\mathbb{R})}\,dt = A_p(\phi)\|f\|_{L^p(\mathbb{R})},$$

where

$$A_p(\phi) = \int_0^\infty |\phi(t)| t^{-1+1/p}\,dt.$$

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  Thus, $\mathcal{H}_\phi$ is bounded in $L^p(\mathbb{R})$, $1 \le p \le \infty$, provided $A_p(\phi) < \infty$.
- Notice that the above simple argument for using Minkowski's inequality cannot be applied to $H^p(\mathbb{R})$ with $p < 1$.
- We shall simply say that $\mathcal{H}_\phi$ is bounded in $H^p(\mathbb{R})$ if $\mathcal{H}_\phi$ is well-defined in a dense subspace of $H^p(\mathbb{R})$ and if it is extended to a bounded operator in $H^p(\mathbb{R})$.

Results

- **Theorem A.** (Kanjin) Let $0 < p < 1$ and $M = [1/p - 1/2] + 1$. Suppose $A_1(\phi) < \infty$, $A_2(\phi) < \infty$, and suppose $\widehat{\phi}$ (the Fourier transform of the function ϕ extended to the whole real line by setting $\phi(t) = 0$ for $t \leqq 0$) is a function of class C^{2M} on $\mathbb{R}$ with $\sup_{\xi \in \mathbb{R}} |\xi|^M |\widehat{\phi}^{(M)}(\xi)| < \infty$ and $\sup_{\xi \in \mathbb{R}} |\xi|^M |\widehat{\phi}^{(2M)}(\xi)| < \infty$. Then $\mathcal{H}_\phi$ is bounded in $H^p(\mathbb{R})$.

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- **Theorem B.** (L-Miyachi) Let $0 < p < 1$, $M = [1/p - 1/2] + 1$, and let ϵ be a positive real number. Suppose ϕ is a function of class C^M on $(0, \infty)$ such that

$$|\phi^{(k)}(t)| \leqq \min\{t^{\epsilon}, t^{-\epsilon}\} t^{-1/p-k} \quad \text{for} \quad k = 0, 1, \ldots, M.$$

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- **Theorem D.** (L-Miyachi) There exists a function ϕ on $(0, \infty)$ such that ϕ is bounded, $\operatorname{supp} \phi$ is a compact subset of $(0, \infty)$, and, for every $p \in (0, 1)$, the operator $\mathcal{H}_\phi$ is not bounded in $H^p(\mathbb{R})$.

Special atomic decomposition - Miyachi

Definition. Let $0 < p \leqq 1$ and let M be a positive integer. For $0 < s < \infty$, we define $\mathcal{A}_{p,M}(s)$ as the set of all those $f \in L^2(\mathbb{R}^n)$ for which $\widehat{f}(\xi) = 0$ for $|\xi| \leqq \frac{1}{s}$ and

$$\|D^\alpha \widehat{f}\|_{L^2} \leqq s^{|\alpha| - \frac{n}{p} + \frac{n}{2}}, \qquad |\alpha| \leq M.$$

We define $\mathcal{A}_{p,M}$ as the union of $\mathcal{A}_{p,M}(s)$ over all $0 < s < \infty$.

Lemma. Let $0 < p \leqq 1$ and M be a positive integer satisfying $M > \frac{n}{p} - \frac{n}{2}$. Then there exists a constant $c_{p,M}$, depending only on n, p and M, such that the following hold.

(1) $\|f(\cdot - x_0)\|_{H^p(\mathbb{R}^n)} \leqq c_{p,M}$ for all $f \in \mathcal{A}_{p,M}$ and all $x_0 \in \mathbb{R}^n$;

(2) Every $f \in H^p(\mathbb{R}^n)$ can be decomposed as $f = \sum_{j=1}^\infty \lambda_j f_j(\cdot - x_j)$, where $f_j \in \mathcal{A}_{p,M}$, $x_j \in \mathbb{R}^n$, $0 \leqq \lambda_j < \infty$, and $\left(\sum_{j=1}^\infty \lambda_j^p\right)^{\frac{1}{p}} \leqq c_{p,M} \|f\|_{H^p(\mathbb{R}^n)}$, and the series converges in $H^p(\mathbb{R}^n)$.

If $f \in H^p \cap L^2$, then this decomposition can be made so that the series converges in L^2 as well.

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- $$\int_{\mathbb{R}^n} |u|^{-n}\Phi(u)f(\frac{x}{|u|})\,du$$

  is but indeed is not bounded in any $H^p(\mathbb{R}^n)$ with $p < 1$.

More general operators

Before proceeding to the multivariate case, consider a somewhat more advanced one-dimensional version of the Hausdorff operator, apparently first introduced by Kuang:

$$(\mathcal{H}f)(x) = (\mathcal{H}_{\varphi,a}f)(x) = \int_{\mathbb{R}_+} \frac{\varphi(t)}{a(t)} f(\frac{x}{a(t)})\, dt\,,$$

where $a(t) > 0$ and $a'(t) > 0$ for all $t \in \mathbb{R}_+$ except maybe $t = 0$.

Theorem E. Let $0 < p < 1$, $M = [1/p - 1/2] + 1$, and let ϵ be a positive real number. Suppose φ is a function of class C^M on $(0, \infty)$ such that φ and a satisfy the compatibility condition

$$\left| \left(\frac{1}{a'(t)} \frac{d}{dt} \right)^k \frac{\varphi(t)}{a'(t)} \right| \leqq \min\{|a(t)|^\epsilon, |a(t)|^{-\epsilon}\} |a(t)|^{-1/p-k}$$

for $k = 0,\, 1,\, \ldots,\, M$. Then $\mathcal{H}_\varphi, a$ is a bounded linear operator in H^p.

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- The Fourier transform of $\mathcal{H}_{\Phi,A} f$ is (formally) calculated from the definition as

$$(\mathcal{H}_{\Phi,A} f)^{\wedge}(\xi) = \int_{\mathbb{R}^N} \Phi(u) \widehat{f}(\xi A(u))\, du, \quad \xi \in \mathbb{R}^n. \tag{1}$$

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- To be precise, we have to put some conditions on Φ, A, and f so that $\mathcal{H}_{\Phi,A} f$ is well-defined and the formula (1) holds.

We give preliminary argument concerning the definition of $\mathcal{H}_{\Phi,A}$ and formula (1).
For functions $\Phi : \mathbb{R}^N \to \mathbb{C}$, $A : \mathbb{R}^N \to M_n(\mathbb{R})$, and $f : \mathbb{R}^n \to \mathbb{C}$, consider

$$(\mathcal{H}_{\Phi,A}f)(x) = \int_{\mathbb{R}^N} \Phi(u)|\det A(u)|^{-1} f(x\,{}^tA(u)^{-1})\,du, \quad x \in \mathbb{R}^n,$$

and

$$(\widetilde{\mathcal{H}}_{\Phi,A}f)(x) = \int_{\mathbb{R}^N} \Phi(u) f(xA(u))\,du, \quad x \in \mathbb{R}^n.$$

We always assume that Φ, A, and f are Borel measurable functions.
Defining

$$L_A(\Phi) = \int_{\mathbb{R}^N} |\Phi(u)|\,|\det A(u)|^{-1/2}\,du,$$

we have the following.

Proposition. If $L_A(\Phi) < \infty$, then for all $f \in L^2(\mathbb{R}^n)$ the functions $\mathcal{H}_{\Phi,A} f$ and $\widetilde{\mathcal{H}}_{\Phi,A} f$ are well-defined almost everywhere on $\mathbb{R}^n$ and the inequalities

$$\|\mathcal{H}_{\Phi,A} f\|_{L^2(\mathbb{R}^n)} \le L_A(\Phi)\|f\|_{L^2(\mathbb{R}^n)}$$

and

$$\|\widetilde{\mathcal{H}}_{\Phi,A} f\|_{L^2(\mathbb{R}^n)} \le L_A(\Phi)\|f\|_{L^2(\mathbb{R}^n)}$$

hold. Thus $\mathcal{H}_{\Phi,A}$ and $\widetilde{\mathcal{H}}_{\Phi,A}$ are well-defined bounded operators in $L^2(\mathbb{R}^n)$ if $L_A(\Phi) < \infty$.

The next proposition gives the formula (1).

Proposition. If $L_A(\Phi) < \infty$, then $(\mathcal{H}_{\Phi,A} f)^\wedge = \widetilde{\mathcal{H}}_{\Phi,A} \widehat{f}$ for all $f \in L^2(\mathbb{R}^n)$.

Guess

- On account of Theorems C and D, one may suppose that the multidimensional operator $\mathcal{H}_{\Phi,A}$ is bounded in $H^p(\mathbb{R}^n)$, $0 < p < 1$, if one merely assumes Φ and A to be sufficiently smooth and Φ to be with compact support.

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- However, this naive generalization of Theorem C is false. There are examples of smooth Φ with compact support and smooth A for which $\mathcal{H}_{\Phi,A}$ is not bounded in the Hardy space $H^p(\mathbb{R}^n)$, $0 < p < 1$, $n \geq 2$.

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- This leads to conclusion that A, or Φ, or both of them should be subject to additional assumptions. The nature and type of such assumptions is, in a sense, the main issue, or, say, spirit of our work.
- Indeed, for positive results, we introduce an algebraic condition on A and prove the Hardy space boundedness of $\mathcal{H}_{\Phi,A}$. This is a generalization of Theorem C to the multidimensional case.

Multidimensional result

Theorem. Let $n \in \mathbb{N}$, $n \geq 2$, $0 < p < 1$, and $M = [n/p - n/2] + 1$. Let $N \in \mathbb{N}$, $\Phi : \mathbb{R}^N \to \mathbb{C}$ be a function of class C^M with compact support, and $A : \mathbb{R}^N \to M_n(\mathbb{R})$ be a mapping of class C^{M+1}. Assume the matrix $A(u)$ is nonsingular for all $u \in \mathrm{supp}\Phi$. Also assume Φ and A satisfy the following condition:

$$\begin{cases} \text{for all } (u, y, \xi) \in \mathrm{supp}\Phi \times \Sigma^{n-1} \times \Sigma^{n-1}, \\ \text{there exists a } j = j(u, y, \xi) \in \{1, \ldots, N\} \text{ such that} \\ \left\langle y, \xi \dfrac{\partial A(u)}{\partial u_j} \right\rangle \neq 0, \end{cases} \tag{2}$$

where $\Sigma = \Sigma^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Then the operator $\mathcal{H}_{\Phi,A}$ is bounded in $H^p(\mathbb{R}^n)$.

Condition in dimension two

- $u = (u_1, u_2)$ $\qquad \partial_j := \frac{\partial}{\partial u_j} \quad j = 1, 2$

$$\frac{\partial A(u)}{\partial u_j} = \begin{pmatrix} \partial_j a_{11}(u) & \partial_j a_{12}(u) \\ \partial_j a_{21}(u) & \partial_j a_{22}(u) \end{pmatrix}$$

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- Condition: for some j

$$\left\langle (\cos y, \sin y), (\cos \xi, \sin \xi) \frac{\partial A(u)}{\partial u_j} \right\rangle$$

$$= \Big\langle (\cos y, \sin y), (\cos \xi \partial_j a_{11}(u) + \sin \xi \partial_j a_{21}(u),$$

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$$= \cos y \cos \xi \partial_j a_{11}(u) + \cos y \sin \xi \partial_j a_{21}(u)$$

$$+ \sin y \cos \xi \partial_j a_{12}(u) + \sin y \sin \xi \partial_j a_{22}(u)$$

Examples – unbounded

Example

Let Φ be a nonnegative smooth function on $(0,\infty)$ with compact support. Assume $\Phi(s) > 1$ for $1 < s < 2$. Then, for $n \geq 2$ and $0 < p < 1$, the operator $(Hf)(x) = \int_0^\infty \Phi(s) f(sx)\, ds, \quad x \in \mathbb{R}^n$, is not bounded in $H^p(\mathbb{R}^n)$.

Let $SO(n,\mathbb{R})$ be the Lie group of real $n \times n$ orthogonal matrices with determinant 1 and let μ be the Haar measure on $SO(n,\mathbb{R})$.

Example

For $n \geq 2$ and $0 < p < 1$, the operator

$$(Hf)(x) = \int_{SO(n,\mathbb{R})} f(xP)\, d\mu(P), \quad x \in \mathbb{R}^n,$$

is not bounded in $H^p(\mathbb{R}^n)$.

Example – bounded

Example below should be compared with the preceding examples; the difference is only more dimensions for averaging but the result is quite opposite.

Example

Let $n \in \mathbb{N}$, $n \geq 2$, $0 < p < 1$, and $M = [n/p - n/2] + 1$. Let $\Phi : (0, \infty) \times SO(n, \mathbb{R}) \to \mathbb{C}$ be a function of class C^M with compact support. Then the operator

$$(Hf)(x) = \int_{(0,\infty)\times SO(n,\mathbb{R})} \Phi(s, P) f(sxP) \, ds d\mu(P), \quad x \in \mathbb{R}^n,$$

is bounded in $H^p(\mathbb{R}^n)$.

Dimensions

We give some remarks concerning the number N in the condition (2). To simplify notation, we write $B_j = \frac{\partial A(u)}{\partial u_j}$. Thus $B_1, \ \ldots, B_N$ are $n \times n$ real matrices.

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$$\begin{cases} \text{for all } (y, \xi) \in \Sigma^{n-1} \times \Sigma^{n-1}, \text{ there exists a } j \in \{1, \ldots, N\} \\ \text{such that } \langle y, \xi B_j \rangle \neq 0. \end{cases} \tag{3}$$

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- The following statement is valid.
  Proposition. (1) *The condition* (3) *is possible only if* $N \geq n$.
  (2) *If* n *is odd and* $n \geq 3$, *then* (3) *is possible only if* $N \geq n+1$.
  (3) *For all* $n \geq 2$, *the condition* (3) *is possible with* $N = 1 + n(n-1)/2$. *If* $n \geq 4$, *then* (3) *is possible with an* $N < 1 + n(n-1)/2$.

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