Interpolating real polynomials

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Interpolating sequences

Let X be a set and H a reproducing kernel Hilbert space of real functions defined on X, i.e. for all $x \in X$, there is a $K_x \in H$ such that

$$f(x) = \langle f, K_x \rangle.$$

We normalize the reproducing kernel and denote $\kappa_x = K_x / \|K_x\|$.

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Definition

A sequence $\Lambda \subset X$ is an interpolating sequence for H whenever

$$\sum_{\lambda \in \Lambda} |c_\lambda|^2 \simeq \| \sum_{\lambda \in \Lambda} c_\lambda \kappa_\lambda \|^2.$$

Riesz sequences and Interpolating sequences in PW

Let $\Lambda \subset \mathbb{R}$, then

Definition

A sequence of functions $\{f_\lambda(z) = \frac{\sin \pi(z-\lambda)}{\pi(z-\lambda)}\}_{\lambda \in \Lambda}$ is a Riesz sequence for the Paley Wiener space whenever,

$$\sum_{\lambda \in \Lambda} |c_\lambda|^2 \lesssim \left| \sum_{\lambda \in \Lambda} c_\lambda f_\lambda \right|^2 \lesssim \sum_{\lambda \in \Lambda} |c_\lambda|^2.$$

This implies that Λ is uniformly separated.

The density of a interpolating sequences

There is a density that almost describes interpolating sequences

Definition

The upper Beurling-Landau density of a sequence $\Lambda \subset \mathbb{R}$ is

$$D^+(\Lambda) = \lim_{r\to\infty} \sup_{x\in\mathbb{R}} \frac{\#\{\Lambda \cap (x-r, x+r)\}}{2r}.$$

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Theorem (Beurling)

A separated sequence $\Lambda \subset \mathbb{R}$ is interpolating for PW if $D^+(\Lambda) < 1$. Moreover if Λ is interpolating then $D^+(\Lambda) \leq 1$.

Our setting

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Let dV be the normalized Lebesgue measure restricted to Ω.
We denote by $N_n = dim(\mathcal{P}_n)$.
We endow $\mathcal{P}_n$ with the norm given by $L^2(V)$.

$$\|p\|^2 = \int_\Omega |f(x)|^2 \, dV(x).$$

Interpolating sequences

Let $\Lambda = \{\Lambda_n\}_n \subset \Omega$ be a sequence of finite sets of points of $\Omega \subset \mathbb{R}^d$.

Definition

We say that Λ is an interpolating sequence if there is a constant $C > 0$ such that

$$C^{-1} \sum_{\lambda \in \Lambda_n} |c_\lambda|^2 \leq \left| \sum_{\lambda \in \Lambda} c_\lambda \kappa_\lambda^n \right|^2 \leq C \sum_{\lambda \in \Lambda_n} |c_\lambda|^2,$$

were κ_λ^n is the normalized reproducing kernel.

We are interested in the geometric distribution of points in Λ.

Alternative definition

Λ is an interpolating is equivalent to the two following properties.

$$\sum_{\lambda\in\Lambda_n} \frac{|p(\lambda)|^2}{K_n(\lambda,\lambda)} \leq C \left\|p\right\|^2, \qquad \forall p \in \mathcal{P}_n$$

and for any sequence of sets of values $\{v_\lambda\}_{\lambda\in\Lambda_v}$ there are polynomials $p_n \in \mathcal{P}_n$ such that $p_n(\lambda) = v_\lambda$ with

$$\|p_n\|^2 \leq C \sum_{\lambda\in\Lambda_n} \frac{|v_\lambda|^2}{K_n(\lambda,\lambda)}.$$

The “natural” normalization

The natural normalization is

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This can be computed as follows. Take $p_1, \ldots, p_{N_n}$ an orthonormal basis of $\mathcal{P}_n$ and construct:

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$$c_{\lambda,n} = K_n(\lambda, \lambda) \simeq \min\left(\frac{n^d}{\sqrt{d(\lambda)}}, n^{d+1} \right).$$

Moreover K_n is the reproducing kernel:

$$p(z) = \int_\Omega K_n(z, w) p(w)\, dV(w), \qquad \forall p \in \mathcal{P}_n$$

Carleson mesures

The Plancherel-Polya sequences are a particular case of Carleson measures.

Definition

A sequence of measures in Ω, μ_k is Carleson if there is a constant $C > 0$ such that

$$\int_\Omega |p|^2 \, d\mu_k \leq C \|p\|^2, \qquad \forall p \in \mathcal{P}_k.$$

We have a geometric characterization of Carleson measures.

An anisotropic metric

In the ball there is an anisotpric distance given by

$$d(x,y) = \arccos\left\{\langle x,y\rangle + \sqrt{1-|x|^2} + \sqrt{1-|y|^2}\right\}.$$

This is the geodesic distance of the points in the sphere S^d defined as $x' = (x, \sqrt{1-|x|^2})$ and $y' = (x, \sqrt{1-|x|^2})$.

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If we consider balls $B(x,r)$ in this distance they are comparable to a box (a product of intervals) which is of size R in the tangent directions and $R^2 + R\sqrt{1-|x|^2}$ in the normal direction.

Geometric characterization

The geometric characterization of the Carleson measures is the following:

Theorem

Let Ω be a ball. A sequence of measures μ_n is Carleson if there is a constant C such that for all points $z \in \Omega$

$$\mu_n(B(z, 1/n)) \leq CV(B(z, 1/n)).$$

Bochner-Riesz type kernels

Proof.

The main ingredient in the proof is the existence of well localized kernels (the needlets of Petrushev and Xu), i.e. kernels $L_n(x,y)$ such that for an arbitrary k there is a constant C_k such that:

$$|L_n(x,y)| \leq C_k \frac{\sqrt{K_n(x,x)K_n(y,y)}}{(1+nd(x,y))^k},$$

and moreover $L_n(x,x) \simeq K_n(x,x)$ and $L_n \in \mathcal{P}_{2n}$ and reproduce the polynomials of degree n.

□

The Nyquist density

We try to identify which is the critical density. We will use the following result:

Theorem (Berman, Boucksom, Witt-Nyström)

If μ is a Bernstein-Markov measure then

$$\frac{K_n(x,x)d\mu(x)}{N_n} \overset{*}{\rightharpoonup} \mu^{eq}.$$

The Bernstein-Markov condition is technical and it is satisfied when $\mu = \chi_\Omega dV$. The measure μ^{eq} is the equilibrium measure.

The equilibrium potential

Definition

Given a compact $K = \overline{\Omega} \subset \mathbb{R}^d$ and any $z \in \mathbb{C}^d$ one defines the Siciak-Zaharjuta equilibrium potential as

$$u_K(z) = \sup\left\{ \frac{\log |p(z)|}{\deg(p)} : \sup_K |p| \leq 1 \right\}.$$

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Then the equilibrium measure is defined as the Monge-Ampere of u_K

$$\mu^{eq} = (i\partial\bar{\partial} u_K)^d.$$

The equilibrium measure is a positive measure supported on K.

What does μ^{eq} look like?

The measure μ^{eq} is a well-known object in pluripotential theory. In the examples we mentioned before it is well understood.

Theorem (Bedford-Taylor)

If Ω is an open bounded convex set in $\mathbb{R}^d$ then

$$d\mu^{eq}(x) \simeq d_{euc}(x, \partial\Omega)^{-1/2} dV(x).$$

Main result

Theorem

If Λ is an interpolating sequence for the polynomials in a bounded smooth strictly convex domain then

$$\limsup_{n\to\infty} \frac{1}{N_n} \sum_{\lambda\in\Lambda_n} \delta_\lambda \leq \mu^{eq}.$$

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In particular, given any ball B in Ω we have

$$\limsup_{n\to\infty} \frac{\#(\Lambda_n \cap B)}{N_n} \leq \mu^{eq}(B),$$

thus μ^{eq} is the Nyquist density.

The Kantorovich-Wasserstein distance

Given a compact metric space K we defines the K-W distance between two measures μ and ν supported in K as

$$KW(\mu,\nu) = \inf_{\rho} \iint_{K\times K} d(x,y)\,d\rho(x,y),$$

where ρ is an admissible measure, i.e. the marginals of ρ are μ and ν respectively.

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$$KW(\mu,\nu) = \inf_{\rho} \iint_{K\times K} d(x,y)d|\rho|(x,y),$$

where ρ is an admissible complex measure, i.e. the marginals of ρ are μ and ν respectively

The complex transport plan

The K-W distance metrizes the weak-$*$ convergence. We want to prove that

$$KW(b_n, \sigma_n) \to 0,$$

where $b_n \leq K_n(x,x)dV(x)/N_n$ is smaller than the Bergman measure and

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The transport plan ρ_n that is convenient to estimate is:

$$\rho_n(x,y) = \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \delta_\lambda(y) \times g_\lambda(x) \frac{K_n(\lambda, x)}{\sqrt{K_n(\lambda,\lambda)}} \, dV(x),$$

where g_λ is the biorthogonal basis to $\left\{ \frac{K_n(\lambda,x)}{\sqrt{K_n(\lambda,\lambda)}} \right\}_{\lambda \in \Lambda_n}$ in the space $\mathcal{F}_n \subset \mathcal{P}_n$ spanned by $\{\kappa_\lambda, \ \lambda \in \Lambda_n\}$

The complex transport plan

The two marginals of ρ_n are

- $\nu_n := \frac{1}{N_n}\mathcal{K}_n(x,x)\,dV(x) \leq \frac{1}{N_n}K_n(x,x)\,dV(x) \overset{*}{\rightharpoonup} \mu^{eq}$

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and

$$KW(\nu_n,\sigma_n) \leq \frac{1}{N_n}\sum_{\lambda\in\Lambda_n}\int_\Omega d(\lambda,x)|g_\lambda(x)|\frac{|K_n(\lambda,x)|}{\sqrt{K_n(\lambda,\lambda)}}\,dV(x).$$

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Thus

$$KW^2(\nu_n,\sigma_n) \lesssim \frac{1}{N_n}\iint d^2(x,y)|K_n(x,y)|^2\,dV(x)\,dV(y).$$

An off-diagonal estimate

Given a bounded function f on M we denote by T_f be the Toeplitz operator on $\mathcal{P}_n \cap L^2(\Omega)$ with symbol f, i.e. $T_f := \Pi_n \circ f \cdot$ where Π_n denotes the orthogonal projection from $L^2(\Omega)$ to $\mathcal{P}_n$.

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$$\mathrm{Tr}\, T_f^2 - \mathrm{Tr}\, T_{f^2} = \frac{1}{2}\int_{\Omega\times\Omega} (f(x) - f(y))^2 \, |K_n(x,y)|^2 \, dV(x)dV(y).$$

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Now, setting $f := x_i$ we observe than on $\mathcal{P}_{n-1}$, $T_f(p) = x_i p$. Therefore $T_{f^2} - T_f^2 = 0$ on $\mathcal{P}_{n-2}$. Therefore:

$$\operatorname{Tr} T_f^2 - \operatorname{Tr} T_{f^2} = O(k^{n-1})$$

and

$$KW^2(\nu_n, \sigma_n) \lesssim \frac{1}{n}.$$

Some extensions

There are many extensions of this result. Of special interest:
Let M be a compact smooth algebraic variety in $\mathbb{R}^m$.
We endow the space of polynomials $\mathcal{P}_n$ restricted to M with the L^2 norm with respect to the Lebesgue measure. We define interpolating sequences Λ as before.

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Theorem

If Λ is an interpolating sequence for the polynomials then

$$\limsup_{n\to\infty} \frac{1}{N_n} \sum_{\lambda\in\Lambda_n} \delta_\lambda \le \mu^{eq}.$$

The equilibrium measure in this setting is comparable to the Lebesgue measure.