Multi-tiling and equidecomposability of polytopes by lattice translates

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Multi-tiling

Let $A \subset \mathbb{R}^d$ be a polytope and denote χ_A as its indicator function. A is said to *multi-tile* by translations with respect to a discrete set $L \subset \mathbb{R}^d$ if

$$A + L := \sum_{\lambda \in L} \chi_A(x - \lambda) = \chi_A * \delta_L(x) = k \quad \text{a.e.}$$

for some positive integer k.

In this talk we always assume L is a full rank lattice and consider the following problem:

Given a polytope A and a lattice L, formulate in effective terms a condition which is necessary and sufficient for the translates of A along L to be a multi-tiling.

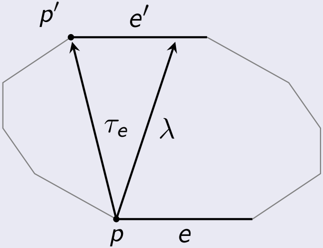
What is known - in the plane

Theorem (Bolle's theorem, 1994)

Let A be a convex polygon in $\mathbb{R}^2$, and L be a lattice in $\mathbb{R}^2$. Then A multi-tiles with respect to L, if and only if A is centrally symmetric, and for each pairing of parallel edges e and e' of A the following two conditions are satisfied:

(i) *for some $\lambda \in L$, both $e + \lambda$ and e' lie on the same line;*

(ii) *if the vector $\vec{e}$ does not belongs to L, then τ_e is in L.*



This theorem was extended by Kolountzakis (2000) to polytopes with the pair property: for each edge of the polygon there is precisely one other edge parallel to it.

What is known - partial results in all dimensions

Necessary conditions:

- Vol(A)$=k\det(L)$, where $\det(L)$ denotes the volume of any fundamental parallelepiped of the lattice L.
- If a convex polytope A is a multi-tiler w.r.t. a discrete set (not necessarily a lattice), then A must be centrally symmetric, have centrally symmetric facets (Gravin, Robins, Shiryaev, 2012).

Sufficient conditions:

- A is centrally symmetric, have centrally symmetric facets, and that all the vertices of A lie in L (Gravin, Robins, Shiryaev, 2012)

Equidecomposability of polytopes - Hilbert's third problem

Hilbert's third problem

Given any two polyhedra in $\mathbb{R}^3$ of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled (by rigid motions) to yield the second?

In $\mathbb{R}^2$, the answer of this problem is "yes" and had been known since 1830s.

Shortly after Hilbert announced this problem, it was solved by his student Dehn, who proved a cube and a regular tetrahedron of equal volume are not equidecomposable under rigid motions.

Additive invariants

Definition

Let G be a subgroup of rigid motions of $\mathbb{R}^d$. A function φ, defined on the set of all polytopes in $\mathbb{R}^d$, is said to be an *additive G-invariant* if

(i) it is additive, namely, if A and B are two polytopes with disjoint interiors then $\varphi(A \cup B) = \varphi(A) + \varphi(B)$;

(ii) it is invariant under motions of the group G, that is, $\varphi(A) = \varphi(g(A))$ whenever A is a polytope and $g \in G$.

One example of additive invariants under rigid motions is the volume function.

Notice if two polytopes A and B are G-equidecomposable, a necessary condition is that $\varphi(A) = \varphi(B)$ for any additive G-invariant φ.

Dehn invariant

In Dehn's solution, he constructed an additive invariant in $\mathbb{R}^3$ with respect to the group of all rigid motions, while a regular tetrahedron and a cube of the same volume take different values under this invariant. Therefore they are not equidecomposable under rigid motions.

Later it was shown by Sydler (1965) that the equality of Dehn invariant is also sufficient for equidecomposability under rigid motions. We say Dehn's invariant is "complete" for equidecomposability with respect to rigid motions.

Hadwiger invariant

If, instead of rigid motions, we only consider translations, the problem of equidecomposability was first introduced in the plane by Hadwiger around 1953.

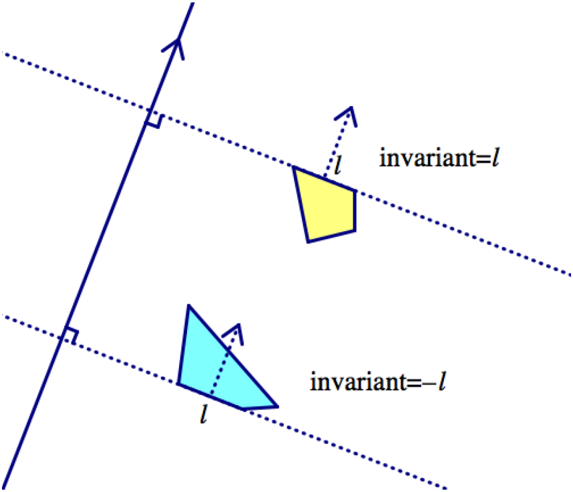
Definition (Hadwiger invariant)

Suppose A is a polygon in the plane and $v \in S^1$ is a direction. Define

$$H_v(A) = \sum_e \epsilon(e)\, length(e),$$

where the sum is taken over all edges e of A that are perpendicular to v. Here $\epsilon(e) = +1$ if the direction v is pointing outward from A at e, and -1 if pointing inward.

Hadwiger invariant



This figure is from an online note of Inna Zakharevich's lectures, transcribed by Elden Elmanto and Henry Chan.

Additive invariants

Theorem (Hadwiger-Glur, 1952)

Two polygons A, B in the plane are equidecomposable under translations if and only if

$$H_v(A) = H_v(B)$$

for any $v \in S^1$.

Similarly one can define Hadwiger invariants in higher dimensions. It is known that Hadwiger invariants are complete in any dimension ($d = 3$, Hadwiger (1968); $d \geq 1$, Jessen-Thorup (1978), Sah (1979) independently).

Multitiling vs Equidecomposablity under lattice translations

If we consider equidecomposability with respect to lattice translations, it is closely related to multi-tiling.

Lemma

Let $A \subset \mathbb{R}^d$ be a polytope and $L \subset \mathbb{R}^d$ be a lattice. Then $A + L$ is a multi-tiling of level k, if and only if A is equidecomposable to a disjoint union of k fundamental domains of L.

In fact we can consider equidecomposability with respect to any proper subgroup of translations. This question was first raised by S.Grepstad and N.Lev in 2014. They also defined Hadwiger-type invariants with respect to any proper subgroup of translations. However they did not prove the "completeness" of these invariants.

Hadwiger-type functionals over lattices

For simplicity, we first introduce our Hadwiger-type functional in the plane.

- Rank 1: Given a line l in the plane and a direction $v \in S^1$ that is perpendicular to l, define

$$H_{l,v}(A, L) = \sum_{e} \epsilon(e)\, length(e),$$

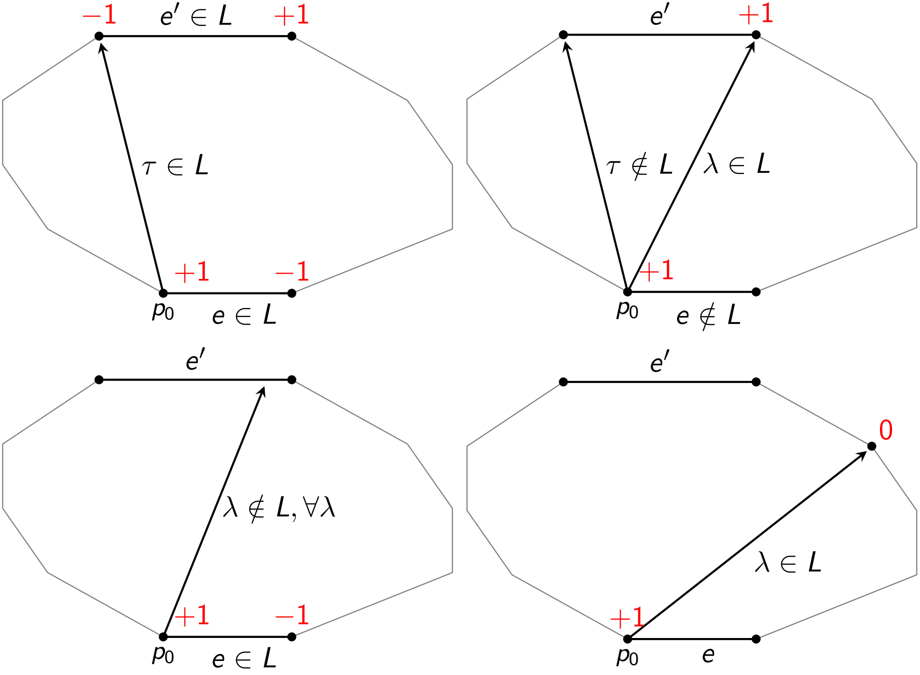
where the sum is taken over all edges e of A that there exists $\lambda \in L$ such that $e + \lambda$ is contained in l. Here $\epsilon(e) = +1$ if the outer normal of e coincides with v, and -1 otherwise.

Hadwiger-type functionals over lattices

- Rank 0: Given a line l and a point $p_0 \in l$, together with a direction $v \perp l$ and a direction $v' // l$, define

$$H_{l,v,p,v'}(A, L) = \sum_{e} \epsilon(e) \sum_{p \in e} \epsilon(e, p)$$

  where the sum in e and $\epsilon(e)$ are as above, and the sum in $p \in e$ is taken over all endpoints $p \in e$ that there exists $\lambda \in L$ such that $p + \lambda = p_0$. Here $\epsilon(e, p) = +1$ if the direction of the vector $\vec{e}$, with p as initial point, coincides with v', and -1 otherwise.



−1
e′ ∈ L
+1
τ ∈ L
+1
−1
p0
e ∈ L
e′
+1
τ ∉ L
λ ∈ L
+1
p0
e ∉ L
e′
λ ∉ L, ∀λ
+1
−1
p0
e ∈ L
e′
0
λ ∈ L
+1
p0
e

In the plane

Theorem (Lev, L., 2018, special case $d = 2$)

Given a full rank lattice L in the plane, a polygon A in the plane multi-tiles under translations from L if and only if

$$H_{l,v}(A, L) = 0, \;\; H_{l,v,p,v'}(A, L) = 0$$

for any line l, any pair (l, p) where $p \in l$ and any choice of directions $v \perp l$, $v' // l$.

Remarks:

- If A is convex, our theorem recovers Bolle's theorem. If A has the pairing property, our theorem recovers Kolountzakis's theorem.
- Our criteria stops after finitely many steps.
- Our proof also works in higher dimensions. In $\mathbb{R}^3$, one can choose any two dimensional hyperplane V, any line $l \subset V$, any point $p \in l \subset V$ and define Hadwiger-type functionals of rank $2, 1, 0$ respectively.

The three dimensional analog of Bolle's theorem

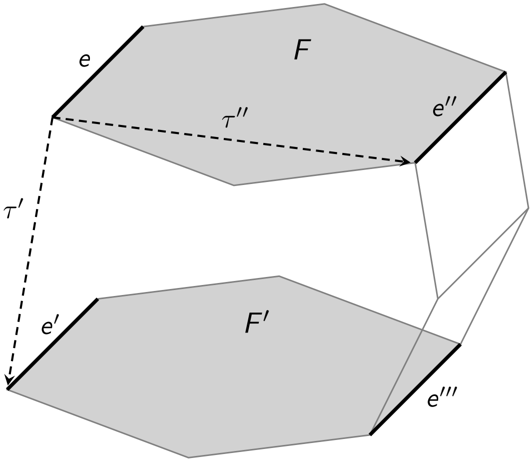


Figure: A four-legged-frame (e, e', e'', e''') of a convex polytope in $\mathbb{R}^3$ that is centrally symmetric and has centrally symmetric facets.

Higher dimensions

Given a polytope $A \subset \mathbb{R}^d$, one can choose a sequence of affine subspaces

$$V_r \subset V_{r+1} \subset \cdots \subset V_d = \mathbb{R}^d,$$

and define Hadwiger-type invariants similarly.

Theorem (Lev, L., 2018)

Given a full rank lattice L in $\mathbb{R}^d$, $d \geq 1$, a polytope A in $\mathbb{R}^d$ multi-tiles under translations from L if and only if $H_*(A, L) = 0$ for any sequence of affine subspaces of $\mathbb{R}^d$.

Back to equidecomposability

As we explained above, under lattice translations, multi-tiling and equidecomposability are closely related. We also prove the following.

Theorem (Lev, L., 2018)

Given a full rank lattice L in $\mathbb{R}^d$, two polytopes A, B in $\mathbb{R}^d$ are equidecomposable under translations from L if and only if they have the same volume and $H_*(A, L) = H_*(B, L)$ for any sequence of of affine subspaces of $\mathbb{R}^d$.

Hint: A, B are equidecomposable with respect to a lattice L if and only if

$$\sum_{\lambda \in L} \chi_A(x - \lambda) = \sum_{\lambda \in L} \chi_B(x - \lambda) \quad \text{a.e.,}$$

which is equivalent to $\sum_{\lambda \in L} (\chi_A - \chi_B)(x - \lambda) = 0$ almost everywhere.

Thank you!