Open problems in wavelet theory

Marcin Bownik

University of Oregon, USA

Frame Theory and Exponential Bases

June 4–8, 2018

ICERM, Brown University, Providence, RI

Meyer wavelets MB (2000)

Suppose ψ is an orthonormal wavelet such that ψ belongs to the Schwartz class. Is $\hat{\psi}(\xi)$ necessarily compactly supported?

Riesz wavelet for H^2 K. Seip ($<$ 1990)

Does there exist a Riesz wavelet ψ for

$$H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \quad \text{for } \xi \leq 0\}$$

such that ψ belongs to the Schwartz class?

Minimality of MSF wavelets D. Larson (1995)

Is it true that for any orthonormal wavelet $\psi \in L^2(\mathbb{R})$, there exists an MSF wavelet ψ_0 such that such that $\operatorname{supp} \hat{\psi}_0 \subset \operatorname{supp} \hat{\psi}$?

Connectivity of wavelets D. Larson and G. Weiss independently (< 1995)

Is the collection of all orthonormal wavelets (or Parseval wavelets or Riesz wavelets) in $L^2(\mathbb{R})$ pathwise connected in $L^2(\mathbb{R})$ norm?

Density of Riesz wavelets D. Larson (1995)

Is the collection of all Riesz wavelets dense in $L^2(\mathbb{R})$?

Intersection of negative dilates L. Baggett (1999)

For a Parseval wavelet ψ define spaces

$$V_i(\psi) = \overline{\text{span}}\{\psi_{j,k} : j < i, k \in \mathbb{Z}\}, \qquad i \in \mathbb{Z}.$$

Is it true that that

$$\bigcap_{j \in \mathbb{Z}} V_j(\psi) = \{0\}.$$

Simple question that nobody has bothered to answer

MB, Weber (2003)

For what values of $\pi/4 < b \leq \pi/3$, is ψ_b a frame wavelet, where $\hat{\psi}_b = \mathbf{1}_{(-2\pi,-b)\cup(b,2\pi)}$?

Simple question that nobody has bothered to answer

MB, Weber (2003)

For what values of $\pi/4 < b \leq \pi/3$, is ψ_b a frame wavelet, where $\hat{\psi}_b = \mathbf{1}_{(-2\pi,-b)\cup(b,2\pi)}$?

Range of b	Property of ψ_b	Duals of ψ_b	$V_0(\psi_b)$
$b = 0$	not a frame wavelet	no duals exist	not SI
$0 < b \leq \pi/4$	frame wavelet (not Riesz)	no affine duals exist	SI
$\pi/3 < b < 2\pi/3$	not a frame wavelet	no duals exist	SI
$2\pi/3 \leq b < \pi$	biorthogonal Riesz wavelet	canonical affine dual exists (=biorthogonal Riesz wavelet)	SI
$b = \pi$	orthonormal wavelet	canonical affine dual exists (=orthonormal wavelet)	SI
$\pi < b \leq 2\pi$	not a frame wavelet	no duals exist	SI

Extension of wavelet frames O. Christensen (2013)

Suppose ψ is Bessel wavelet with bound < 1. Does there exist ψ_1 such that the wavelet system generated by ψ and ψ_1 is a Parseval wavelet?

Characterization of dilations D. Speegle (1997)

For what dilations $A \in GL_n(\mathbb{R})$ and lattices $\Gamma \subset \mathbb{R}^n$, there exist an orthonormal wavelet (or an MSF wavelet) associated with (A, Γ)?

Calderón's formula D. Speegle (2001)

Does Calderón's formula

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}((A^T)^j\xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n$$

hold for orthonormal (or Parseval) wavelets associated with (A, Γ)?

Schwartz class wavelets for integer dilations MB, Speegle (2001)

Do Schwartz class wavelets exist for integer expansive dilations A and lattice $\Gamma = \mathbb{Z}^n$?

Well-localized wavelets Daubechies (1992)

For what expansive dilations do there exist well-localized wavelets (possibly with multiple generators)?