Riesz bases, Meyer's quasicrystals, and bounded remainder sets

Sigrid Grepstad

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Joint work with Nir Lev

Riesz bases of exponentials

$S \subset \mathbb{R}^d$ is bounded and measurable.

$\Lambda \subset \mathbb{R}^d$ is discrete.

The exponential system

$$E(\Lambda) = \{e_\lambda\}_{\lambda \in \Lambda}, \quad e_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle},$$

is a Riesz basis in the space $L^2(S)$ if the mapping

$$f \to \{\langle f, e_\lambda \rangle\}_{\lambda \in \Lambda}$$

is bounded and invertible from $L^2(S)$ onto $\ell^2(\Lambda)$.

Known results

Kozma, Nitzan (2012): Finite unions of intervals

Kozma, Nitzan (2015): Finite unions of rectangles in $\mathbb{R}^d$

G., Lev and Kolountzakis (2012/2013): Multi-tiling sets in $\mathbb{R}^d$

Lyubarskii, Rashkovskii (2000): Convex, centrally symmetric polygons in $\mathbb{R}^2$

Questions

- What about the ball in dimensions two and higher?
- Does every set in $\mathbb{R}^d$ admit a Riesz basis of exponentials?

Density

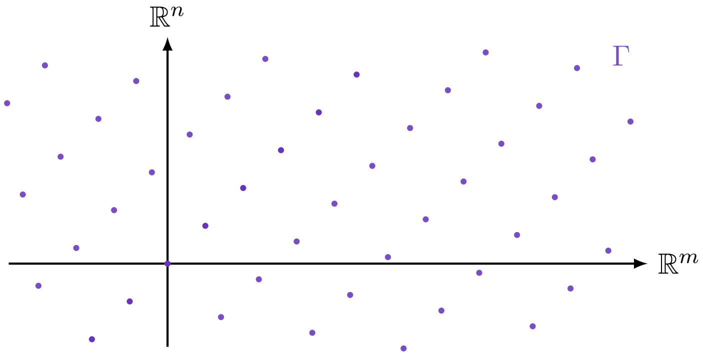
Lower and upper uniform densities:

$$D^-(\Lambda) = \liminf_{R\to\infty} \inf_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap (x + B_R))}{|B_R|}$$

$$D^+(\Lambda) = \limsup_{R\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap (x + B_R))}{|B_R|}$$

If $E(\Lambda)$ is a Riesz basis in $L^2(S)$, then $D^-(\Lambda) = D^+(\Lambda) = \operatorname{mes} S$.

Cut-and-project sets



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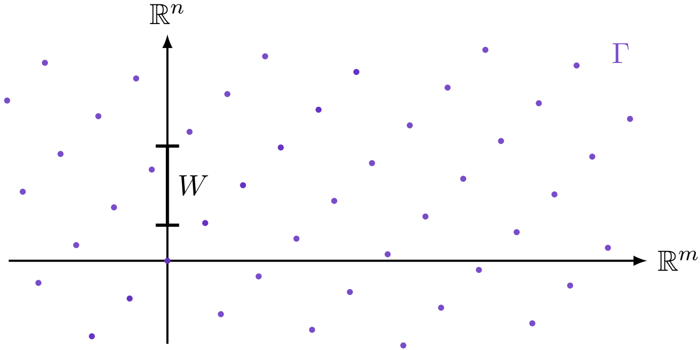

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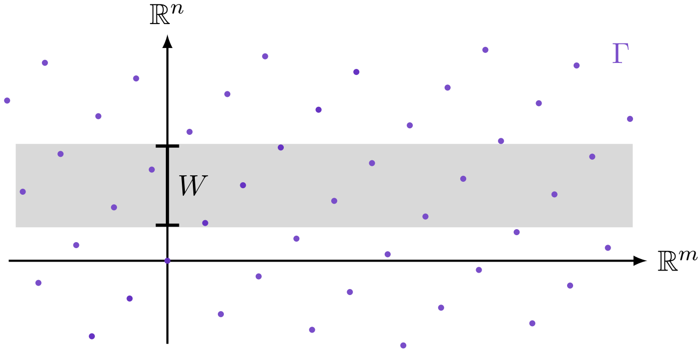

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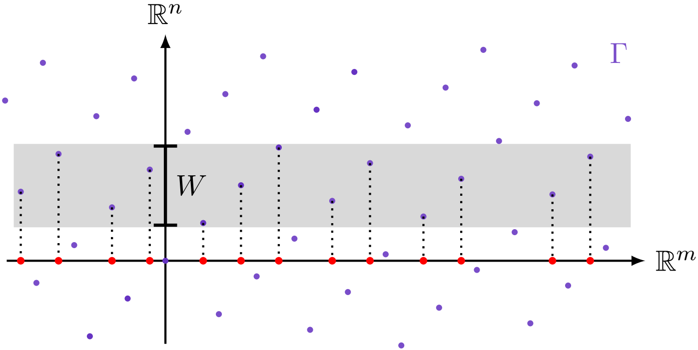

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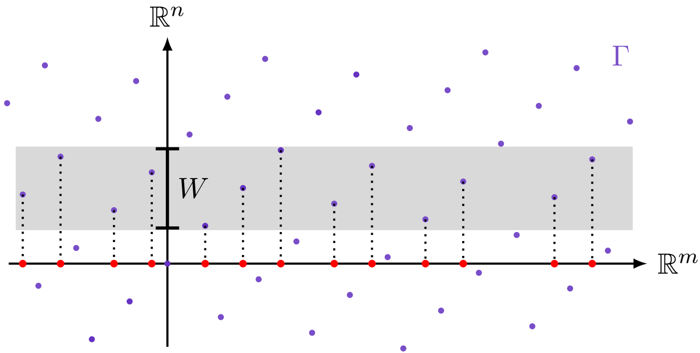
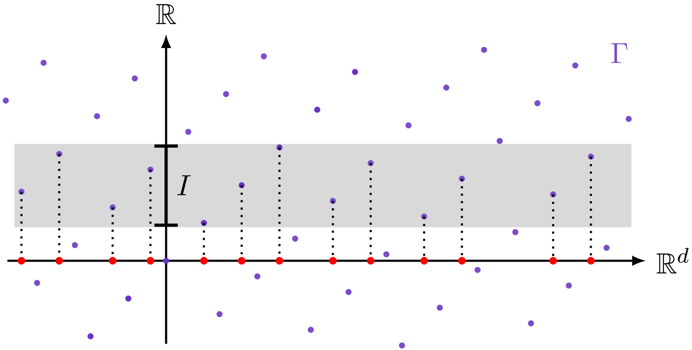


We define the Meyer cut-and-project set

$$\Lambda(\Gamma, W) = \{p_1(\gamma) \,:\, \gamma \in \Gamma,\, p_2(\gamma) \in W\},$$

with density $D(\Lambda) = \operatorname{mes} W / \det \Gamma$.

Simple quasicrystals



We define the simple quasicrystal

$$\Lambda(\Gamma, I) = \{p_1(\gamma) \,:\, \gamma \in \Gamma,\, p_2(\gamma) \in I\},$$

with density $D(\Lambda) = |I|/\det\Gamma$.

Sampling on quasicrystals

Matei and Meyer (2008): Simple quasicrystals are universal sampling sets.

Kozma, Lev (2011): Riesz bases of exponentials from quasicrystals in dimension one.

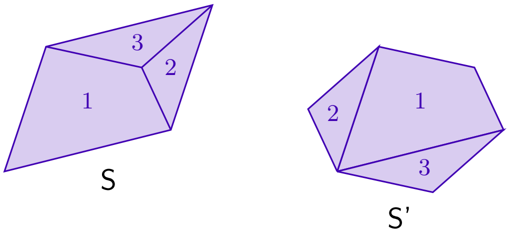
Theorem 1

Let $\Lambda = \Lambda(\Gamma, I)$, and suppose that

$$|I| \notin p_2(\Gamma).$$

Then there exists no Riemann measurable set S such that $E(\Lambda)$ is a Riesz basis in $L^2(S)$

Equidecomposability



The sets S and S' are equidecomposable (or scissors congruent).

Theorem 2

Let $\Lambda = \Lambda(\Gamma, I)$, and suppose that

$$|I| \in p_2(\Gamma).$$

Then $E(\Lambda)$ is a Riesz basis in $L^2(S)$ for every Riemann measurable set S, $\operatorname{mes} S = D(\Lambda)$, satisfying the following condition:

S is equidecomposable to a parallelepiped with vertices in $p_1(\Gamma^*)$, using translations by vectors in $p_1(\Gamma^*)$.

$$\Gamma^* = \left\{\gamma^* \in \mathbb{R}^d \times \mathbb{R} \,:\, \langle \gamma, \gamma^* \rangle \in \mathbb{Z} \text{ for all } \gamma \in \Gamma\right\}$$

Example 1

Let α be an irrational number, and define $\Lambda = \{\lambda(n)\}$ by

$$\lambda(n) = n + \{n\alpha\}, \quad n \in \mathbb{Z}.$$

Then $E(\Lambda)$ is a Riesz basis in $L^2(S)$ for every $S \subset \mathbb{R}$, $\operatorname{mes} S = 1$, which is a finite union of disjoint intervals with lengths in $\mathbb{Z}\alpha + \mathbb{Z}$.

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Notice that $\{\lambda(n)\}_{n\in\mathbb{Z}} = \Lambda(\Gamma, I)$, where $I = [0, 1)$ and

$$\Gamma = \{((1+\alpha)n - m, n\alpha - m) \, : \, m, n \in \mathbb{Z}\},$$
$$\Gamma^* = \{(n\alpha + m, -n(1+\alpha) - m) \, : \, m, n \in \mathbb{Z}\}$$

Example 2

Let $\Lambda = \{\lambda(n, m)\}$ be defined by

$$\lambda(n, m) = (n, m) + \{n\sqrt{2} + m\sqrt{3}\}(\sqrt{2}, \sqrt{3}), \quad (n, m) \in \mathbb{Z}^2.$$

$E(\Lambda)$ is a Riesz basis in $L^2(S)$ for every set $S \subset \mathbb{R}^2$ which is equidecomposable to the unit cube $[0, 1)^2$ using only translations by vectors in $\mathbb{Z}(\sqrt{2}, \sqrt{3}) + \mathbb{Z}^2$.

Corollary 1

$\Lambda = \Lambda(\Gamma, I)$, $|I| \in p_2(\Gamma)$

$K \subset \mathbb{R}^d$ compact, $U \subset \mathbb{R}^d$ open

$K \subset U$ and $\operatorname{mes} K < D(\Lambda) < \operatorname{mes} U$

There exists a set $S \subset \mathbb{R}^d$ satisfying:

i) $K \subset S \subset U$ and $\operatorname{mes} S = D(\Lambda)$.

ii) S is equidecomposable to a parallelepiped with vertices in $p_1(\Gamma^*)$ using translations by vectors in $p_1(\Gamma^*)$.

Duality

$$\Lambda(\Gamma, I) = \{p_1(\gamma) \,:\, \gamma \in \Gamma,\, p_2(\gamma) \in I\} \subset \mathbb{R}^d$$
$$\Lambda^*(\Gamma, S) = \{p_2(\gamma^*) \,:\, \gamma^* \in \Gamma^*,\, p_1(\gamma^*) \in S\} \subset \mathbb{R}$$

Duality lemma

Suppose that $E(\Lambda^*(\Gamma, S))$ is a Riesz basis in $L^2(I)$. Then $E(\Lambda(\Gamma, I))$ is a Riesz basis in $L^2(S)$.

Lattices of special form

$$\Gamma = \left\{ \Big((\mathrm{Id} + \beta\alpha^\top)m - \beta n, n - \alpha^\top m \Big) \ : \ m \in \mathbb{Z}^d, \, n \in \mathbb{Z} \right\}$$

$$\Gamma^* = \left\{ \Big(m + n\alpha, (1 + \beta^\top\alpha)n + \beta^\top m \Big) \ : \ m \in \mathbb{Z}^d, \, n \in \mathbb{Z} \right\}$$

Theorem 2

Let $\Lambda = \Lambda(\Gamma, I)$ and suppose that

$$|I| = m_1\alpha_1 + \cdots m_d\alpha_d + n$$

for integers $m_1, \ldots, m_d$ and n. Then $E(\Lambda)$ is a Riesz basis in $L^2(S)$ for every Riemann measurable set S, $\operatorname{mes} S = |I|$, which is equidecomposable to a parallelepiped with vertices in $\mathbb{Z}^d + \alpha\mathbb{Z}$ using translations by vectors in $\mathbb{Z}^d + \alpha\mathbb{Z}$.

By duality, we may choose to consider

$$\Lambda^*(\Gamma, S) = \Big\{ n + \beta^\top (n\alpha + m) \,:\, n\alpha + m \in S \Big\},$$

where $n \in \mathbb{Z}$ and $m \in \mathbb{Z}^d$.

Question: When is $E(\Lambda^*)$ a Riesz basis in $L^2(I)$ for an interval of length $|I| = \operatorname{mes} S$?

Avdonin's theorem

Avdonins theorem

Let $I \subset \mathbb{R}$ be an interval and $\Lambda = \{\lambda_j : j \in \mathbb{Z}\}$ be a sequence in $\mathbb{R}$ satisfying:

(a) Λ is separated;

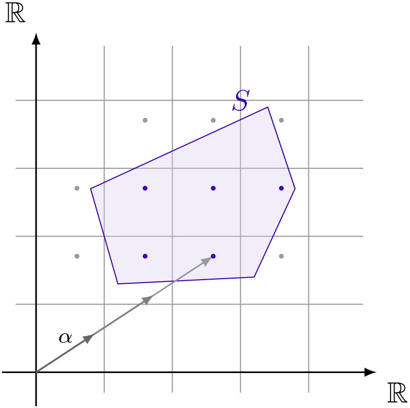
(b) $\sup_j |\delta_j| < \infty$, where $\delta_j := \lambda_j - j/|I|$;

(c) There is a constant c and positive integer N such that

$$\sup_{k \in \mathbb{Z}} \left| \frac{1}{N} \sum_{j=k+1}^{k+N} \delta_j - c \right| < \frac{1}{4|I|}$$

Then $E(\Lambda)$ is a Riesz basis in $L^2(I)$.

$$\Lambda^*(\Gamma, S) = \left\{ n + \beta^\top (n\alpha + m) \,:\, n \in \mathbb{Z},\, m \in \mathbb{Z}^d,\, n\alpha + m \in S \right\}$$
$$= \bigcup \Lambda_n, \quad \Lambda_n = \left\{ n + \beta^\top s \,:\, s = n\alpha + m \in S \right\}$$

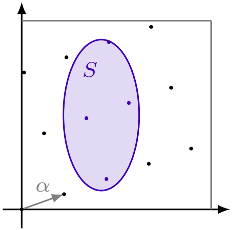


Irrational rotation on the torus

$S \subset \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$

The sequence $\{n\alpha\}$ is equidistributed.



$$\frac{1}{n} \sum_{k=0}^{n-1} \chi_S(x + k\alpha) \to \operatorname{mes} S \quad (n \to \infty)$$

$$D_n(S, x) = \sum_{k=0}^{n-1} \chi_S(x + k\alpha) - n \operatorname{mes} S = o(n)$$

Bounded remainder sets

Definition

A set S is a *bounded remainder set* (BRS) if there is a constant $C = C(S, \alpha)$ such that

$$|D_n(S,x)| = \left| \sum_{k=0}^{n-1} \chi_S(x + k\alpha) - n \operatorname{mes} S \right| \leq C$$

for all n and a.e. x.

Claim: The quasicrystal $\Lambda^*(\Gamma, S)$ is at bounded distance from $\{j/\operatorname{mes} S\}_{j\in\mathbb{Z}}$ if and only if S is a bounded remainder set.

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$$= \bigcup_n \Lambda_n\,, \quad \Lambda_n = \Big\{ n + \beta^\top s \,:\, s = n\alpha + m \in S \Big\}$$

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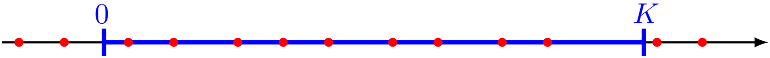

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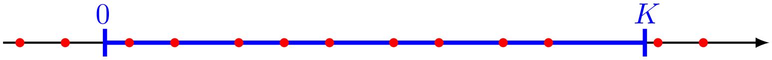

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$$N = |\Lambda^* \cap [0,K)| = \sum_{k=0}^{K-1} |\Lambda_k| + const = \sum_{k=0}^{K-1} \chi_S(k\alpha) + const$$

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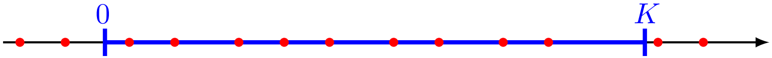

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$$= |\mathbb{Z}/\operatorname{mes} S \cap [0, K)| + const = K \operatorname{mes} S + const$$

Properties of bounded remainder sets

Theorem (G., Lev 2015)

Any parallelepiped in $\mathbb{R}^d$ spanned by vectors $v_1, \dots, v_d$ belonging to $\mathbb{Z}\alpha + \mathbb{Z}^d$ is a bounded remainder set.

(Duneau and Oguey (1990): *Displacive transformations and quasicrystalline symmetries*)

Theorem

The measure of any bounded remainder set must be of the form

$$n_0 + n_1\alpha_1 + \cdots + n_d\alpha_d$$

where $n_0, \dots n_d$ are integers.

Characterization of Riemann measurable BRS

Theorem

A Riemann measurable set $S \subset \mathbb{R}^d$ is a BRS if and only if there is a parallelepiped P spanned by vectors belonging to $\mathbb{Z}\alpha + \mathbb{Z}^d$, such that S and P are equidecomposable using translations by vectors in $\mathbb{Z}\alpha + \mathbb{Z}^d$ only.

Summary proof Theorem 2

$\Lambda^*(\Gamma, S)$ provides a Riesz basis $E(\Lambda^*)$ in $L^2(I)$ whenever $S \subset \mathbb{R}^d$ is a bounded remainder set with $\operatorname{mes} S = |I|$, i.e. if S is equidecomposable to a parallelepiped spanned by vectors in $\mathbb{Z}\alpha + \mathbb{Z}^d$ using translations by vectors in $\mathbb{Z}\alpha + \mathbb{Z}^d$.

$\Downarrow$ (Duality)

$\Lambda(\Gamma, I)$ gives a Riesz basis $E(\Lambda)$ in $L^2(S)$ for all such sets S.

Note: The given equidecomposition condition on S implies that

$$\operatorname{mes} S = n_0 + n_1\alpha_1 + \cdots + n_d\alpha_d \in p_2(\Gamma).$$

Pavlov's complete characterization

One can deduce from Pavlov's complete characterization of exponential Riesz bases in $L^2(I)$ that for $\Lambda^* = \Lambda^*(\Gamma, S)$ to provide a Riesz basis in $L^2(I)$ it is necessary that the sequence of discrepancies

$$\{d_n\}_{n\geq 1} = \left\{ \sum_{k=0}^{n-1} \chi_S(k\alpha) - n \operatorname{mes} S \right\}_{n\geq 1}$$

is in BMO, i.e. satisfies

$$\sup_{n<m} \left(\frac{1}{m-n} \sum_{k=n+1}^{m} \left| d_k - \frac{d_{n+1} + \cdots + d_m}{m-n} \right| \right) < \infty.$$

Theorem (Kozma and Lev, 2011)

If the sequence

$$\left\{\sum_{k=0}^{n-1} \chi_S(k\alpha) - n \operatorname{mes} S\right\}_{n\geq 1}$$

belongs to BMO, then the measure of S is of the form

$$n_0 + n_1\alpha_1 + \cdots + n_d\alpha_d,$$

where $n_0, n_1, \ldots, n_d$ are integers.

Open problem

Suppose that the condition

$$|I| = n_0 + n_1\alpha_1 + \cdots + n_d\alpha_d$$

is satisfied. Are there additional sets $S \subset \mathbb{R}^d$ which admit $E(\Lambda(\Gamma, I))$ as a Riesz basis?

Related question: Does there exist a set S for which the sequence

$$\left\{ \sum_{k=0}^{n-1} \chi_S(k\alpha) - n \,\mathrm{mes}\, S \right\}_{n\geq 1}$$

is unbounded, but in BMO?

Thank you for your attention.