Uncertainty Principles for Fourier Multipliers

Michael Northington V

School of Mathematics
Georgia Tech

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With Shahaf Nitzan (Ga Tech) and Alex Powell (Vanderbilt)

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- Question 2: Why do we care about this setting?

Example 1: Gabor Systems and the Zak Transform

- Gabor System: For $g \in L^2(\mathbb{R})$,

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- Leads to an isometric isomorphism:
  - $L^2(\mathbb{R}) \to L^2_w(\mathbb{T}^2)$, for $w = |Zg|^2$
  - $G(g) \to E = E(2)$.

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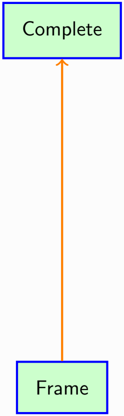
  - $h \to m$
  - $T(f) \to E = E(d)$

Spanning and Independence Properties



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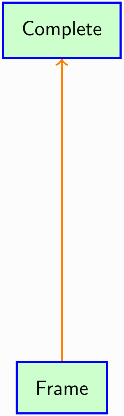

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- H is a frame if it's complete, and there exist constants $0 < A \le B < \infty$ with

$$\forall h \in \mathcal{H}, \quad A\|h\|_{\mathcal{H}}^2 \le \sum_{n=1}^{\infty} |\langle h, h_n \rangle|^2 \le B\|h\|_{\mathcal{H}}^2.$$

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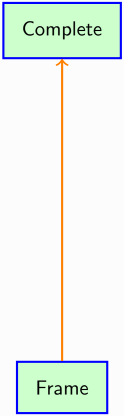

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- Every frame is complete, with the additional bonus that there exist a choice of coefficients such that $h = \sum c_n h_n$ with

$$\|c_n\|_{l^2} \asymp \|h\|_{\mathcal{H}}.$$

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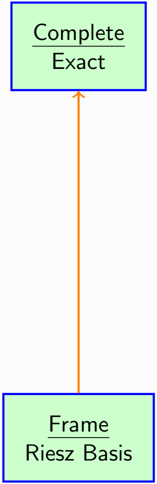

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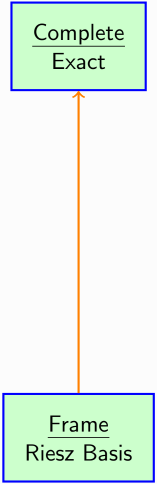

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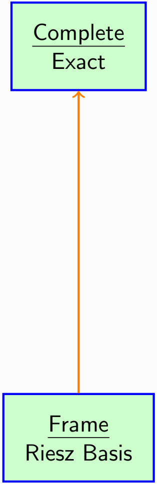

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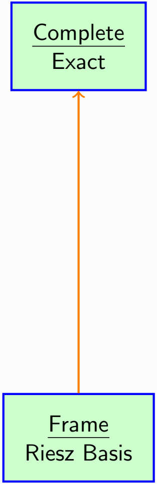

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- Riesz basis $\implies$ frame; Riesz basis $\iff$ minimal frame.

(C_q)-systems (Olevskii, Nitzan '07)

- Fix $2 \leq q \leq \infty$. $\{h_n\}_{n=1}^{\infty} \subset \mathcal{H}$ is a (C_q)-system if for each $h \in \mathcal{H}$, h can be approximated to arbitrary accuracy by a finite sum $\sum a_n h_n$ such that

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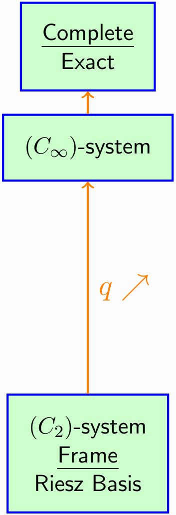
(C_q)-systems


Complete
Exact
(C_∞)-system
$q \nearrow$
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Riesz Basis

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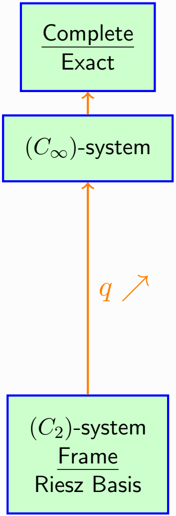
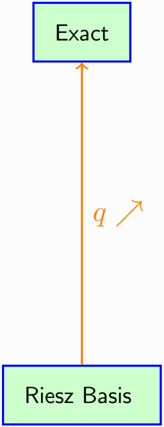


For Exact E in $L^2_w(\mathbb{T}^d)$:



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- Nitzan, Olsen ('11) gave necessary and sufficient conditions similar to the fourth characterization

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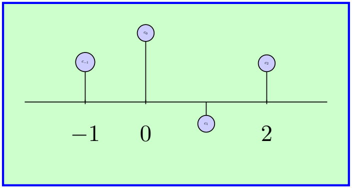

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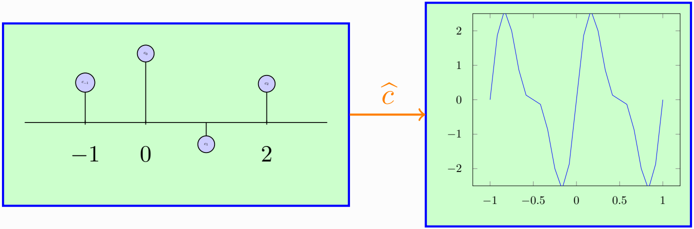

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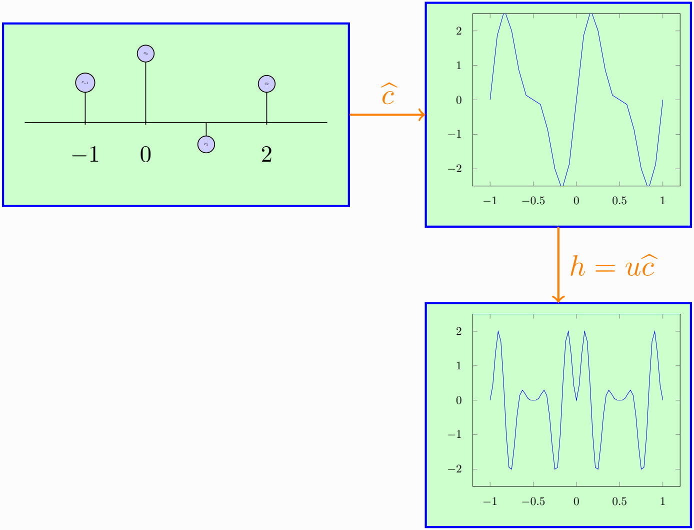

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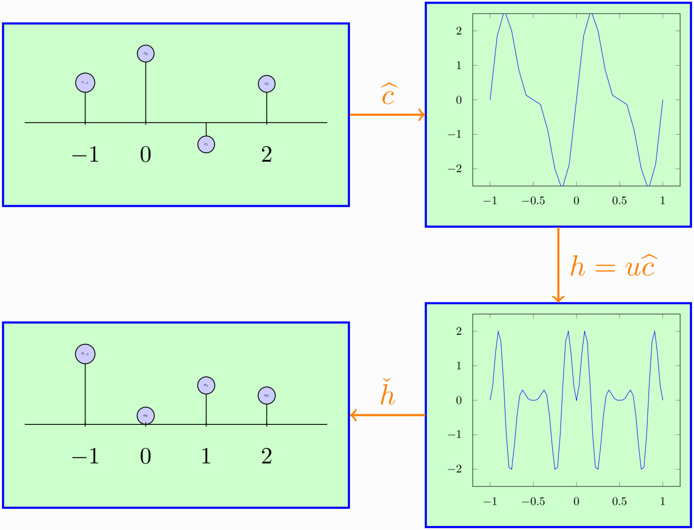

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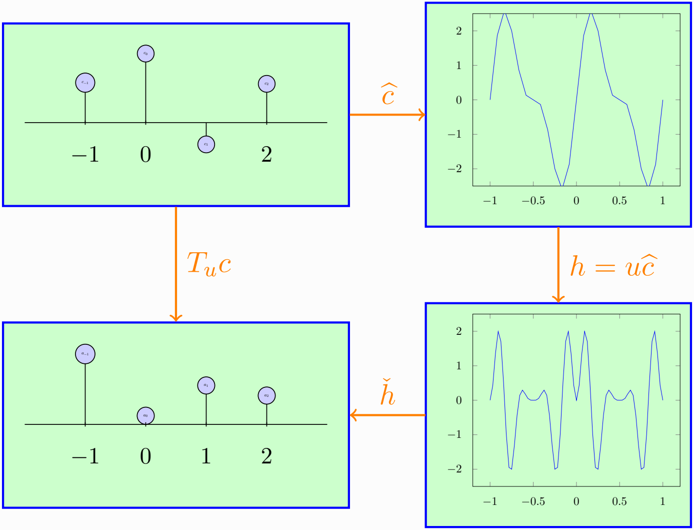

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Sobolev Space:

$$H^s(\mathbb{T}^d) = \{f \in L^2(\mathbb{T}^d) : \sum_{k \in \mathbb{Z}^d} |k|^{2s} |\widehat{f}(k)|^2 < \infty\}$$

Results with a single zero

Theorem (Nitzan, M.N., Powell)

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- Proof relies on Sobolev Embedding Theorem in Hölder spaces.
- For $s > \frac{d}{2} + 1$, we can't say more than the bound in part 2 unless we require a zero of a larger order.

Zero Sets of Larger Hausdorff Dimension

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- A similar question was studied by Jiang, Lin ('03) and Schikorra ('13) with the Fourier multiplier condition replaced with an integrability condition.

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- Proof uses a version of Poincare Inequality from Jiang, Lin ('03) and Schikorra ('13).

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The Balian-Low Theorem

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Let $f \in L^2(\mathbb{R})$*. If* $\mathcal{G}(f) = \{e^{2\pi i m x} f(x-n)\}_{m,n\in\mathbb{Z}}$ *is a Riesz basis for* $L^2(\mathbb{R})$*, then*

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- The Riesz basis property forces $|Zf| \geq A > 0$, which gives contradiction.

Sharp (C_q)-system BLT

Theorem (Nitzan, M.N, Powell)

Fix $q > 2$. If $\mathcal{G}(f,1,1) = \{e^{2\pi i m x} f(x-n)\}_{m,n\in\mathbb{Z}}$ is an exact (C_q)-system for $L^2(\mathbb{R})$, then

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- Follows from single zero multiplier result.
- Nitzan, Olsen ('11) proved similar result, with an additional ϵ on the weight, as well as non-symmetric versions.
- The $q = \infty$ case gives the BLT for exact systems (originally due to Daubechies, Janssen ('93)) and nonsymmetric versions were given by Heil and Powell ('09)

Shift-Invariant Spaces with Extra Invariance

For a given shift-invariant space $V = V(f) \subset L^2(\mathbb{R}^d)$,

- For $\Gamma \subset \mathbb{R}^d$, V is Γ-invariant if $T_\gamma V \subset V$ for all $\gamma \in \Gamma$.

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- Aldroubi, Sun, Wang (2011), and Tessera, Wang (2014), showed that Balian-Low type results exist for shift-invariant spaces with extra-invariance.

(C_q)-system SIS BLT

Theorem (Nitzan, M.N., Powell)

Fix $2 \leq q \leq \infty$. Suppose that $f \in L^2(\mathbb{R})$ is nonzero and $V(f)$ is $\frac{1}{N}\mathbb{Z}$-invariant. If $T(f)$ is a minimal (C_q)-system in $V(f)$, then

$$\int_{\mathbb{R}} |x|^{2(1-1/q)} |f(x)|^2 dx = \infty.$$

Equivalently, $\widehat{f} \notin H^{1-1/q}(\mathbb{R})$.

- If $T(f)$ is a minimal system for $V(f)$, then $T(f)$ is a (C_∞)-system. Thus, the $q = \infty$ case gives us a result for minimal systems.
- (Hardin, M.N., Powell) In the $q = 2$ case, the result holds in higher dimensions, and without assuming minimality. (i.e., frames and not necessarily Riesz bases)

Minimal (C_q)-result Higher Dimensions

Theorem

Fix q such that $2 \leq q \leq \infty$, and let $s = \min(d(\frac{1}{2} - \frac{1}{q}) + \frac{1}{2}, 1)$. Let $0 \neq f \in L^2(\mathbb{R}^d)$, and suppose $V(f)$ is invariant under some non-integer shift. If $\mathcal{T}(f)$ is a minimal (C_q)-system for $V(f)$ then

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- Probably the sharp s is $1 - 1/q$ in all dimensions.

Where does the zero come from?

- Extra-invarance can be characterized in terms of $P\widehat{f}$. (Aldroubi, Cabrelli, Heil, Kornelson, Molter (2010), Anastasio, Cabrelli, Paternostro (2011))

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$$
\begin{aligned}
P(x) &= \sum_{k\in\mathbb{Z}^2} |\widehat{f}(x-k)|^2 \\
&= \sum_{k\in\mathbb{Z}^2} |\widehat{f}(x-2k)|^2 + \sum_{k\in\mathbb{Z}^2} |\widehat{f}(x-2k+e_1)|^2 \\
&\quad + \sum_{k\in\mathbb{Z}^2} |\widehat{f}(x-2k+e_2)|^2 + \sum_{k\in\mathbb{Z}^2} |\widehat{f}(x-2k+e_1+e_2)|^2 \\
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$$

Where does the zero come from?

- Extra-invarance can be characterized in terms of $P\widehat{f}$. (Aldroubi, Cabrelli, Heil, Kornelson, Molter (2010), Anastasio, Cabrelli, Paternostro (2011))
- The condition is somewhat technical, so lets look at an example of $f \in L^2(\mathbb{R}^2)$ and $V(f)$ having $\frac{1}{2}\mathbb{Z}^2$-invariance.

$$\begin{aligned} P(x) &= \sum_{k\in\mathbb{Z}^2} |\widehat{f}(x-k)|^2 \\ &= \sum_{k\in\mathbb{Z}^2} |\widehat{f}(x-2k)|^2 + \sum_{k\in\mathbb{Z}^2} |\widehat{f}(x-2k+e_1)|^2 \\ &\quad + \sum_{k\in\mathbb{Z}^2} |\widehat{f}(x-2k+e_2)|^2 + \sum_{k\in\mathbb{Z}^2} |\widehat{f}(x-2k+e_1+e_2)|^2 \\ &= P_2(x) + P_2(x+e_1) + P_2(x+e_2) + P_2(x+e_1+e_2). \end{aligned}$$

- $V(f)$ is $\frac{1}{2}\mathbb{Z}^2$-invariant iff $P_2(x)$ and it's shifts have disjoint support.

Thanks

Thanks!!!