A New Fractional Process: A Fractional Non-homogeneous Poisson Process

Enrico Scalas

University of Sussex

Joint work with Nikolai Leonenko (Cardiff University) and Mailan Trinh

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Overview

Overview

1. Definitions
2. Limit theorems
3. Application to the CTRW
4. Summary and Outlook

Classification of Poisson processes

	standard	fractional
homogeneous	(i) $(N^h_\lambda(t))$	(iii) $(N^{hf}_\alpha(t))$
inhomogeneous	(ii) $(N(t))$	(iv) $(N_\alpha(t))$

The standard (non-fractional) case

(i) The **homogeneous Poisson process** (HPP) $(N^h_\lambda(t))$ with intensity parameter $\lambda > 0$:

$$p^\lambda_x(t) := \mathbb{P}(N^h_\lambda(t) = x) = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

(ii) The **inhomogeneous Poisson process** (NHPP) $(N(t))$ with intensity $\lambda(t) : [0, \infty) \longrightarrow [0, \infty)$ and rate function

$$\Lambda(s, t) = \int_s^t \lambda(u)\mathrm{d}u.$$

For $x = 0, 1, 2, \ldots$, the distribution of the increment is

$$p_x(t, v) := \mathbb{P}\{N(t + v) - N(v) = x\} = \frac{\mathrm{e}^{-\Lambda(v,t+v)}(\Lambda(v, t + v))^x}{x!}.$$

Note that $N(t) = N^h_1(\Lambda(t))$.

The (inverse) α-stable subordinator

Let $L_\alpha = \{L_\alpha(t), t \geq 0\}$, be an **$\alpha$-stable subordinator** with Laplace transform

$$\mathbb{E}\left[\exp(-sL_\alpha(t))\right] = \exp(-ts^\alpha), \quad 0 < \alpha < 1, s \geq 0$$

and $Y_\alpha = \{Y_\alpha(t), t \geq 0\}$, be an **inverse α-stable subordinator** defined by

$$Y_\alpha(t) = \inf\{u \geq 0 : L_\alpha(u) > t\}.$$

Let $h_\alpha(t, \cdot)$ denote the density of the distribution of $Y_\alpha(t)$.
Its Laplace transform can be expressed via the three-parameter **Mittag-Leffler function** (a.k.a Prabhakar function).

$$\mathbb{E}\left[\exp(-sY_\alpha(t))\right] = E^1_{\alpha,1}(-st^\alpha), \text{ where}$$

$$E^c_{a,b}(z) = \sum_{j=0}^{\infty} \frac{c^{\bar{j}} z^j}{j!\Gamma(aj+b)}, \text{ with}$$

$$c^{\bar{j}} = c(c+1)(c+2)\ldots(c+j-1), a > 0, b > 0, c > 0, z \in \mathbb{C}.$$

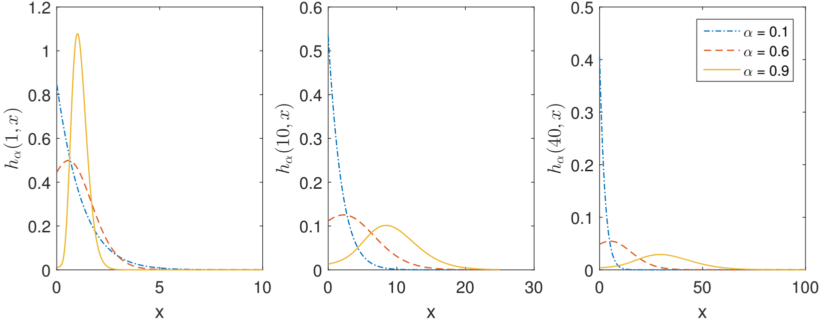

Figure: Plots of the probability densities $x \mapsto h_\alpha(t,x)$ of the distribution of the inverse α-stable subordinator $Y_\alpha(t)$ for different parameter $\alpha = 0.1, 0.6, 0.9$ and as a function of time: the plot on the left is generated for $t = 1$, the plot in the middle for $t = 10$ and the plot on the right for $t = 40$. The x scale is not kept constant.

The fractional case

(iii) The **fractional homogeneous Poisson process** (FHPP) $(N_\alpha^{hf}(t))$ is defined as $N_\alpha^{hf}(t) := N_\lambda^h(Y_\alpha(t))$ for $t \geq 0, 0 < \alpha < 1$. Its marginal distribution is given by

$$p_x^\alpha(t) = \mathbb{P}\{N_\lambda(Y_\alpha(t)) = x\} = \int_0^\infty \mathrm{e}^{-\lambda u} \frac{(\lambda u)^x}{x!} h_\alpha(t,u)\mathrm{d}u$$
$$= (\lambda t^\alpha)^x E_{\alpha,\alpha x+1}^{x+1}(-\lambda t^\alpha),\ x = 0, 1, 2, \ldots$$

(iv) The **fractional non-homogenous Poisson process** (FNPP) could be defined in the following way:
Recall that the NPP can be expressed via the HPP:

$$N(t) = N_1^h(\Lambda(t)).$$

Analogously define $N_\alpha(t) := N(Y_\alpha(t)) = N_1^h(\Lambda(Y_\alpha(t)))$

The governing equation for the FNPP

We can define the marginals

$$f_x^\alpha(t,v) := \mathbb{P}\{N_1^h(\Lambda(Y_\alpha(t)+v)) - N_1^h(\Lambda(v)) = x\}, \quad x = 0,1,2,\ldots$$
$$= \int_0^\infty p_x(u,v) h_\alpha(t,u) \mathrm{d}u$$

Theorem (Leonenko et al. (2017))

Let $I_\alpha(t,v) = N_1^h(\Lambda(Y_\alpha(t)+v)) - N_1^h(\Lambda(v))$ be the fractional increment process. Then, its marginal distribution satisfies the following fractional differential-integral equations ($x = 0,1,\ldots$)

$$D_t^\alpha f_x^\alpha(t,v) = \int_0^\infty \lambda(u+v)[-p_x(u,v) + p_{x-1}(u,v)] h_\alpha(t,u) \mathrm{d}u,$$

with initial condition $f_x^\alpha(0,v) = \delta_0(x)$ and $f_{-1}^\alpha(0,v) \equiv 0$.

Overview

Limit theorems for the Poisson process

Watanabe (1964): The **compensator** of $N_\lambda^h(t)$ is λt, i.e. $N_\lambda^h(t) - \lambda t$ is a martingale. (Watanabe characterisation)

One-dimensional central limit theorem

$$\frac{N_\lambda^h(t) - \lambda t}{\sqrt{\lambda t}} \xrightarrow[t\to\infty]{d} \mathcal{N}(0,1)$$

Functional central limit theorem: convergence in $\mathcal{D}([0,\infty))$ w.r.t. J_1-topology to a standard Brownian motion $(B(t))_{t\geq 0}$.

$$\left(\frac{N_\lambda^h(t) - \lambda t}{\sqrt{\lambda}}\right)_{t\geq 0} \xrightarrow[\lambda\to\infty]{J_1} B$$

Functional **scaling** limit:

$$\left(\frac{N_\lambda^h(ct)}{c}\right)_{t\geq 0} \xrightarrow[c\to\infty]{J_1} (\lambda t)_{t\geq 0}$$

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Random time change and continuous mapping theorem

We have convergence in $\mathcal{D}([0,\infty))$ w.r.t. J_1-topology to a standard Brownian motion $(B(t))_{t\geq 0}$.

$$\left(\frac{N_\lambda^h(t)-\lambda t}{\sqrt{\lambda}}\right)_{t\geq 0} \xrightarrow[\lambda\to\infty]{J_1} B.$$

As B has **continuous** paths and Y_α has **non-decreasing** paths, it follows that

$$\left(\frac{N_\lambda^h(Y_\alpha(t))-\lambda Y_\alpha(t)}{\sqrt{\lambda}}\right)_{t\geq 0} \xrightarrow[\lambda\to\infty]{J_1} \left[B(Y_\alpha(t))\right]_{t\geq 0}.$$

(Thm. 13.2.2 in Whitt (2002))

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Cox processes: definition

Idea: Poisson process with stochastic intensity. (Cox (1955))

- $\rightarrow$ actuarial risk models (e.g. Grandell (1991))
- $\rightarrow$ credit risk models (e.g. Bielecki and Rutkowski (2002))
- $\rightarrow$ filtering theory (e.g. Brémaud (1981))

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(N(t))_{t\geq 0}$ be a point process adapted to $(\mathcal{F}_t^N)_{t\geq 0}$. $(N(t))_{t\geq 0}$ is a Cox process if there exist a right-continuous, increasing process $(A(t))_{t\geq 0}$ such that, conditional on the filtration $(\mathcal{F}_t)_{t\geq 0}$, where

$$\mathcal{F}_t := \mathcal{F}_0 \vee \mathcal{F}_t^N, \quad \mathcal{F}_0 = \sigma(A(t), t \geq 0),$$

$(N(t))_{t\geq 0}$ is a Poisson process with intensity $\mathrm{d}A(t)$.

Cox processes and the FNPP

Is the FNPP a Cox process?

The FHPP is also a renewal process: handy criteria in Yannaros (1994), Grandell (1976), Kingman (1964).

Construction of a suitable filtration: $N_\alpha(t) = N_1^h(\Lambda(Y_\alpha(t)))$.

$$\mathcal{F}_t^{N_\alpha} := \sigma(\{N_\alpha(s), s \leq t\})$$
$$\mathcal{F}_0 := \sigma(Y_\alpha(t), t \geq 0)$$
$$\mathcal{F}_t := \mathcal{F}_0 \vee \mathcal{F}_t^{N_\alpha}$$

A central limit theorem

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$$\mathcal{F}_0 := \sigma(Y_\alpha(t), t \geq 0)$$
$$\mathcal{F}_t := \mathcal{F}_0 \vee \mathcal{F}_t^{N_\alpha}$$

Proposition

Let $(N(Y_\alpha(t)))_{t\geq 0}$ be the FNPP adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ as defined in previous slide. Then,

$$\frac{N(Y_\alpha(T)) - \Lambda(Y_\alpha(T))}{\sqrt{\Lambda(Y_\alpha(T))}} \xrightarrow[T\to\infty]{d} \mathcal{N}(0,1). \tag{1}$$

Proof: apply Thm. 14.5.I. in Daley and Vere-Jones (2008).

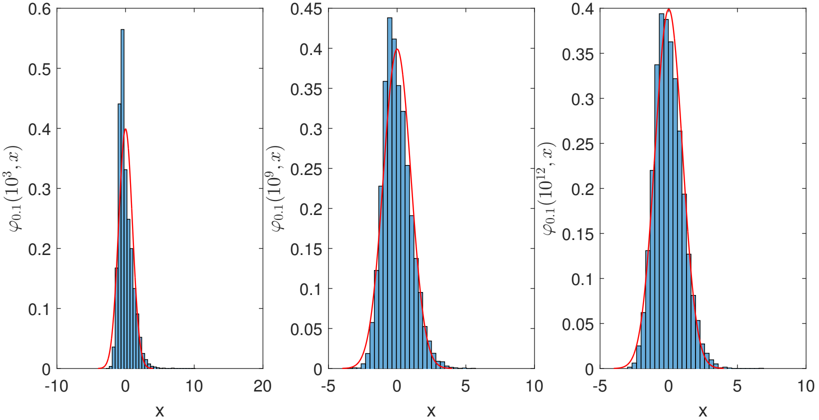

Figure: The red line shows the probability density function of the standard normal distribution, the limit distribution according previous proposition. The blue histograms depict samples of size 10^4 of the right hand side of (1) for different times $t = 10, 10^9, 10^{12}$ for $\alpha = 0.1$ to illustrate convergence to the standard normal distribution.

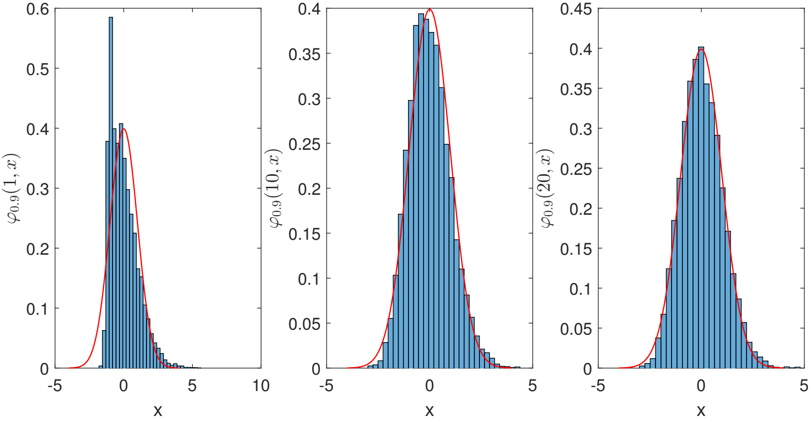

Figure: The red line shows the probability density function of the standard normal distribution, the limit distribution according to previous theorem. The blue histograms depict samples of size 10^4 of the right hand side of (1) for different times $t = 1, 10, 20$ for $\alpha = 0.9$ to illustrate convergence to the standard normal distribution.

Limit $\alpha \to 1$

Proposition

Let $(N_\alpha(t))_{t\geq 0}$ be the FNPP. Then, we have the limit

$$N_\alpha \xrightarrow[\alpha\to 1]{J_1} N \quad \text{in} \quad D([0,\infty)).$$

Idea of the proof: According to Theorem VIII.3.36 on p. 479 in Jacod and Shiryaev (2003) it suffices to show

$$\Lambda(Y_\alpha(t)) \xrightarrow[\alpha\to 1]{\mathcal{P}} \Lambda(t), \quad t \in \mathbb{R}_+$$

By the continuous mapping theorem we need to show

$$Y_\alpha(t) \xrightarrow[\alpha\to 1]{d} t \qquad \forall t \in \mathbb{R}_+.$$

This can be proven by convergence of the respective Laplace transforms:

$$\mathcal{L}\{h_\alpha(\cdot, y)\}(s,y) = E_\alpha(-ys^\alpha) \xrightarrow{\alpha\to 1} \mathrm{e}^{-ys} = \mathcal{L}\{\delta_0(\cdot - y)\}(s,y).$$

Limit theorems for the Poisson process

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Functional central limit theorem: convergence in $\mathcal{D}([0,\infty))$ w.r.t. J_1-topology to a standard Brownian motion $(B(t))_{t\geq 0}$.

$$\left(\frac{N_\lambda^h(t) - \lambda t}{\sqrt{\lambda}}\right)_{t\geq 0} \xrightarrow[\lambda\to\infty]{J_1} B$$

Functional **scaling** limit:

$$\left(\frac{N_\lambda^h(ct)}{c}\right)_{t\geq 0} \xrightarrow[c\to\infty]{J_1} (\lambda t)_{t\geq 0}$$

A scaling limit (one-dimensional limit)

Assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Theorem

Let $(N_\alpha(t))_{t\geq 0}$ be the FNPP. Suppose the function $t \mapsto \Lambda(t)$ is regularly varying with index $\beta > 0$, i.e. for $x \in [0, \infty)$

$$\frac{\Lambda(xt)}{\Lambda(t)} \xrightarrow[t\to\infty]{} x^\beta.$$

Then the following limit holds for the FNPP:

$$\frac{N_\alpha(t)}{\Lambda(t^\alpha)} \xrightarrow[t\to\infty]{d} (Y_\alpha(1))^\beta.$$

A functional scaling limit

Assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Theorem

Let $(N_\alpha(t))_{t\geq 0}$ be the FNPP. Suppose the function $t \mapsto \Lambda(t)$ is regularly varying with index $\beta > 0$, i.e. for $x \in [0, \infty)$

$$\frac{\Lambda(xt)}{\Lambda(t)} \xrightarrow[t\to\infty]{} x^\beta.$$

Then the following limit holds for the FNPP:

$$\left(\frac{N_\alpha(t\tau)}{\Lambda(t^\alpha)}\right)_{\tau\geq 0} \xrightarrow[t\to\infty]{J_1} \left(Y_\alpha(\tau)^\beta\right)_{\tau\geq 0}. \qquad (2)$$

Proof

Using Thm. 2 on p. 81 in Grandell (1976), it suffices to show that

$$\left(\frac{\Lambda(Y_\alpha(t\tau))}{\Lambda(t^\alpha)}\right)_{\tau\geq 0} \xrightarrow[t\to\infty]{J_1} \left(Y_\alpha(\tau)^\beta\right)_{\tau\geq 0}$$

1. **Convergence of finite-dimensional distributions**: By self-similarity of Y_α and Lévy's continuity theorem. (Details in the next slides)
2. **Tightness**: As $\tau \mapsto \Lambda(Y_\alpha(t\tau))$ and $\tau \mapsto Y_\alpha(\tau)$ are continuous and increasing. Thm VI.3.37(a) in Jacod and Shiryaev (2003) ensures tightness.

Proof (II)

Let $t > 0$ be fixed at first, $\tau = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathbb{R}_+^n$ and $\langle \cdot, \cdot \rangle$ denote the scalar product in $\mathbb{R}^n$. Then,

$$\frac{\Lambda(t^\alpha Y_\alpha(\tau))}{\Lambda(t^\alpha)} = \left(\frac{\Lambda(t^\alpha Y_\alpha(\tau_1))}{\Lambda(t^\alpha)}, \frac{\Lambda(t^\alpha Y_\alpha(\tau_2))}{\Lambda(t^\alpha)}, \ldots, \frac{\Lambda(t^\alpha Y_\alpha(\tau_n))}{\Lambda(t^\alpha)} \right) \in \mathbb{R}_+^n$$

Its characteristic function is given by

$$\begin{aligned}
\varphi_t(u) &:= \mathbb{E}\left[\exp\left(\mathrm{i}\left\langle u, \frac{\Lambda(Y_\alpha(t\tau))}{\Lambda(t^\alpha)}\right\rangle\right)\right] = \mathbb{E}\left[\exp\left(\mathrm{i}\left\langle u, \frac{\Lambda(t^\alpha Y_\alpha(\tau))}{\Lambda(t^\alpha)}\right\rangle\right)\right] \\
&= \int_{\mathbb{R}_+^n} \exp\left(\mathrm{i}\left\langle u, \frac{\Lambda(t^\alpha x)}{\Lambda(t^\alpha)}\right\rangle\right) h_\alpha(\tau, x)\mathrm{d}x \\
&= \int_{\mathbb{R}_+^n} \left[\prod_{k=1}^{n} \exp\left(\mathrm{i} u_k \frac{\Lambda(t^\alpha x_k)}{\Lambda(t^\alpha)}\right)\right] h_\alpha(\tau_1, \ldots, \tau_n; x_1, \ldots, x_n)\mathrm{d}x_1 \ldots \mathrm{d}x_n
\end{aligned}$$

Proof (III)

We may estimate

$$\left| \exp\left(\mathrm{i} \left\langle u, \frac{\Lambda(t^\alpha x)}{\Lambda(t^\alpha)} \right\rangle \right) h_\alpha(\tau, x) \right| \leq h_\alpha(\tau, x).$$

By dominated convergence

$$\begin{aligned}
\lim_{t\to\infty} \varphi_t(u) &= \lim_{t\to\infty} \int_{\mathbb{R}_+^n} \exp\left(\mathrm{i} \left\langle u, \frac{\Lambda(t^\alpha x)}{\Lambda(t^\alpha)} \right\rangle \right) h_\alpha(\tau, x) \mathrm{d}x \\
&= \int_{\mathbb{R}_+^n} \lim_{t\to\infty} \exp\left(\mathrm{i} \left\langle u, \frac{\Lambda(t^\alpha x)}{\Lambda(t^\alpha)} \right\rangle \right) h_\alpha(\tau, x) \mathrm{d}x \\
&= \int_{\mathbb{R}_+^n} \exp\left(\mathrm{i} \left\langle u, x^\beta \right\rangle \right) h_\alpha(\tau, x) \mathrm{d}x = \mathbb{E}[\exp(i\langle u, (Y_\alpha(\tau))^\beta \rangle)].
\end{aligned}$$

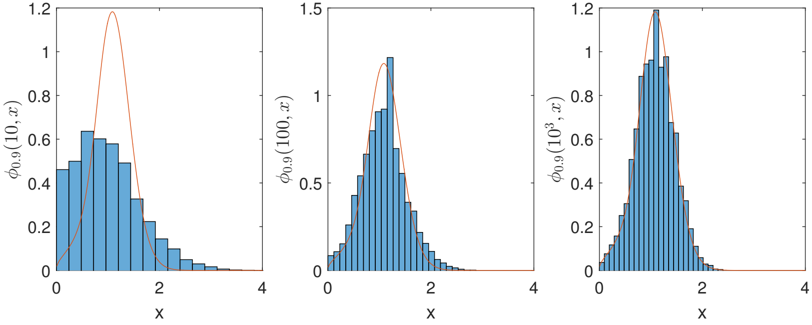

Figure: Red line: probability density function ϕ of the distribution of the random variable $(Y_{0.9}(1))^{0.7}$, the limit distribution according to previous Theorem. The blue histogram is based on 10^4 samples of the random variables on the right hand side of (2) for time points $t = 10, 100, 10^3$ to illustrate the convergence result.

Overview

Proposition (The fractional compound Poisson process)

Let $(N_\alpha(t))_{t\geq 0}$ be the FNPP and suppose the function $t \mapsto \Lambda(t)$ is **regularly varying** *with index $\beta \in \mathbb{R}$. Moreover let $X_1, X_2, \ldots$ be i.i.d. random variables independent of N_α. Assume that the law of X_1 is in the* **domain of attraction of a stable law**, *i.e. there exist sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ and a stable Lévy process $(S(t))_{t\geq 0}$ such that for*

$$\bar{S}_n(t) := a_n \sum_{k=1}^{\lfloor nt \rfloor} X_k - b_n \quad \text{it holds that} \quad \bar{S}_n \xrightarrow[n\to\infty]{J_1} S.$$

Then the fractional compound Poisson process $Z(t) := S_{N_\alpha(t)} = \sum_{k=1}^{N_\alpha(t)} X_k$ has the following limit:

$$(c_n Z(nt))_{t\geq 0} \xrightarrow[n\to\infty]{M_1} \left(S\left([Y_\alpha(t)]^\beta \right) \right)_{t\geq 0},$$

where $c_n = a_{\lfloor \Lambda(n) \rfloor}$.

One-dimensional limit

The previous proposition implies for fixed $t > 0$

$$c_n \sum_{k=1}^{N_\alpha(nt)} X_k \xrightarrow[n\to\infty]{d} S((Y_\alpha(t))^\beta)$$

In the one-dimensional case we can do better:

- We do not need independence between $N(t)$ and $X_1, X_2, \ldots$ (Anscombe (1952))
- Additionally, $X_1, X_2, \ldots$ can be mixing (Mogyoródi (1967), Csörgő and Fischler (1973))

Overview

Summary and Outlook

- We gave a reasonable definition of a fractional non-homogeneous Poisson process that fits into pre-existing theory and results. $\Rightarrow$ Other possible definitions of FNPP: $N_1(Y_\alpha(\Lambda(t)))$
- We derived limit theorems for the FNPP $\Rightarrow$ Parameter estimation
- Other related stochastic processes: Skellam processes, integrated processes

Thank you for your attention!

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