Maximum principle for the fractional diffusion equations and its applications

Yuri Luchko

Department of Mathematics, Physics, and Chemistry
Beuth Technical University of Applied Sciences Berlin
Berlin, Germany

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Outline of the talk:

- Maximum principle for a time-fractional diffusion equation with the general fractional derivative
- Maximum principle for the weak solutions of a time-fractional diffusion equation with the Caputo derivative
- Maximum principle for an abstract space- and time-fractional evolution equation in the Hilbert space
- Short survey of other results

What is a maximum principle?

Maximum principle:
A function satisfies a differential inequality or equation in a domain $D \Rightarrow$
It achieves its maximum on the boundary of D.

A very elementary example:
$f''(x) > 0, \;\; x \in]a, b[$ and $f \in C([a, b]) \Rightarrow$
f achieves its maximum value at one of the endpoints of the interval.

Other examples:
Maximum principles for ordinary differential equations and inequalities
Maximum principles for partial differential equations and inequalities
Very recently: maximum principles for fractional PDEs

General fractional derivatives

Let k be a nonnegative locally integrable function.

The general fractional derivative of the Caputo type:

$$(\mathbb{D}^{C}_{(k)} f)(t) = \int_0^t k(t-\tau) f'(\tau)\, d\tau.$$

The general fractional derivative of the Riemann-Liouville type:

$$(\mathbb{D}^{RL}_{(k)} f)(t) = \frac{d}{dt} \int_0^t k(t-\tau) f(\tau)\, d\tau.$$

For an absolutely continuous function f with the inclusion $f' \in L_1^{loc}(\mathbb{R}_+)$, we get

$$(\mathbb{D}^{C}_{(k)} f)(t) = \frac{d}{dt} \int_0^t k(t-\tau) f(\tau)\, d\tau \; - \; k(t) f(0) \; = \; (\mathbb{D}^{RL}_{(k)} f)(t) - k(t) f(0)$$

Particular cases

1) The conventional Caputo and Riemann-Liouville fractional derivatives:

$$k(\tau) = \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)},\ 0 < \alpha < 1.$$

$$(\mathbb{D}_*^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau)\, d\tau,$$

$$(\mathbb{D}^\alpha f)(t) = \frac{d}{dt} \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f(\tau)\, d\tau \right).$$

Particular cases

2) The multi-term derivatives

$$k(\tau) = \sum_{k=1}^{n} a_k \frac{\tau^{-\alpha_k}}{\Gamma(1-\alpha_k)}, \quad 0 < \alpha_1 < \cdots < \alpha_n < 1$$

$$(\mathbb{D}^{C}_{(k)} f)(t) = \sum_{k=1}^{n} a_k (\mathbb{D}^{\alpha_k}_{*} f)(t),$$

$$(\mathbb{D}^{RL}_{(k)} f)(t) = \sum_{k=1}^{n} a_k (\mathbb{D}^{\alpha_k} f)(t).$$

Particular cases

3) Derivatives of the distributed order:

$$k(\tau) = \int_0^1 \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}\, d\rho(\alpha),$$

where ρ is a Borel measure on $[0, 1]$:

$$(\mathbb{D}^C_{(k)} f) = \int_0^1 (\mathbb{D}^\alpha_* f)(t)\, d\rho(\alpha),$$

$$(\mathbb{D}^C_{(k)} f) = \int_0^1 (\mathbb{D}^\alpha f)(t)\, d\rho(\alpha).$$

Conditions on the kernel function

K1) The Laplace transform $\tilde{k}$ of k,

$$\tilde{k}(p) = \int_0^\infty k(t)\, e^{-pt}\, dt,$$

exists for all $p > 0$,

K2) $\tilde{k}(p)$ is a Stiltjes function,

K3) $\tilde{k}(p) \to 0$ and $p\tilde{k}(p) \to \infty$ as $p \to \infty$,

K4) $\tilde{k}(p) \to \infty$ and $p\tilde{k}(p) \to 0$ as $p \to 0$.

Properties of the general derivatives

(A) For any $\lambda > 0$, the initial value problem for the fractional relaxation equation

$$(\mathbb{D}_{(k)}^{C} f)(t) = -\lambda f(t), \quad t > 0, \quad u(0) = 1$$

has a unique solution $u_\lambda = u_\lambda(t)$ that belongs to the class $C^\infty(\boldsymbol{R}_+)$ and is a completely monotone function.

(B) There exists a completely monotone function $\kappa = \kappa(t)$ with the property

$$\int_0^t k(t-\tau)\kappa(\tau)\, d\tau \quad \equiv 1, \;\; t > 0.$$

(C) For $f \in L_1^{loc}(\boldsymbol{R}_+)$, the relations

$$(\mathbb{D}_{(k)}^{C} \mathcal{I}_{(k)} f)(t) = f(t), \;\; (\mathbb{D}_{(k)}^{RL} \mathcal{I}_{(k)} f)(t) = f(t)$$

hold true, where the general fractional integral $\mathcal{I}_{(k)}$ is defined by the formula

$$(\mathcal{I}_{(k)} f)(t) = \int_0^t \kappa(t-\tau) f(\tau)\, d\tau.$$

General time-fractional diffusion equation

Let Ω be an open and bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ (for example, of C^2 class) and $T > 0$.

The general time-fractional diffusion equation:

$$(\mathbb{D}_{(k)}^{C} u(x,\cdot))(t) = D_2(u) + D_1(u) - q(x)u(x,t) + F(x,t), \ \ (x,t) \in \Omega \times (0,T],$$

where $q \in C(\bar{\Omega})$, $q(x) \geq 0$, $x \in \bar{\Omega}$,

$$D_1(u) = \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}, \ \ D_2(u) = \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}$$

and D_2 is a uniformly elliptic differential operator.

Cauchy problem

Kochubei considered the Cauchy problem with the initial condition

$$u(x,0) = u_0(x), x \in \boldsymbol{R}^n$$

for the homogeneous general time-fractional diffusion equation with $D_2 = \Delta$, $D_1 \equiv 0$ and $q \equiv 0$.

His main results:

1) The Cauchy problem has a unique appropriately defined solution for a bounded globally Hölder continuous initial value u_0.

2) The fundamental solution to the Cauchy problem can be interpreted as a probability density function and thus the general time-fractional diffusion equation describes a kind of (anomalous) diffusion.

Initial-boundary-value problem

Luchko and Yamamoto: Analysis of uniqueness and existence of solution to the initial-boundary-value problem for the general time-fractional diffusion equation with the initial condition

$$u(x,t)\big|_{t=0} = u_0(x),\ \ x \in \bar{\Omega}$$

and the Dirichlet boundary condition

$$u(x,t)\big|_{(x,t)\in\partial\Omega\times(0,T]} = v(x,t),\ \ (x,t) \in \partial\Omega \times (0,T].$$

Their main results:
1) Uniqueness of solution both in the strong and in the weak senses (Estimates of the general fractional derivatives –> Maximum principle for the general diffusion equation –> A priory norm estimates of solutions –> Uniqueness of solutions).
2) Existence of solution in the weak sense (Separation of variables –> Formal solutions in form of generalized Fourier series –> Convergence analysis of the formal solutions).

Estimates of the general fractional derivatives

Let the conditions

L1) $k \in C^1(\boldsymbol{R}_+) \cap L_1^{loc}(\boldsymbol{R}_+)$,

L2) $k(\tau) > 0$ and $k'(\tau) < 0$ for $\tau > 0$,

L3) $k(\tau) = o(\tau^{-1}),\ \tau \to 0.$
be fulfilled.

Let a function $f \in C([0,T])$ attain its maximum over the interval $[0,T]$ at the point $t_0,\ t_0 \in (0,T]$, and $f' \in C(0,T] \cap L_1(0,T)$. Then the following inequalities hold true:

$$(\mathbb{D}_{(k)}^{RL} f)(t_0) \geq k(t_0) f(t_0),$$

$$(\mathbb{D}_{(k)}^{C} f)(t_0) \geq k(t_0)(f(t_0) - f(0)) \geq 0.$$

Maximum principle

Let us define the operator

$$\mathbb{P}_{(k)}(u) := (\mathbb{D}^C_{(k)} f)(t) - D_2(u) - D_1(u) + q(x)u(x,t).$$

Let the conditions L1)-L3) be fulfilled and a function u satisfy the inequality

$$\mathbb{P}_{(k)}(u) \le 0, \ (x,t) \in \Omega \times (0,T].$$

Then the following maximum principle holds true:

$$\max_{(x,t)\in\ \bar{\Omega}\times[0,T]} u(x,t) \le \max\{\max_{x\in\bar{\Omega}} u(x,0), \max_{(x,t)\in\ \partial\Omega\times[0,T]} u(x,t),\ 0\}.$$

A priori norm estimates

Let the conditions K1)-K4) and L1)-L3) be fulfilled and u be a solution of the initial-boundary-value problem for the general diffusion equation.

Then the following estimate of the uniform solution norm holds true:

$$\|u\|_{C(\bar{\Omega}\times[0,T])} \le \max\{M_0,\ M_1\} + M\, f(T),$$

where

$$M_0 = \|u_0\|_{C(\bar{\Omega})},\ M_1 = \|v\|_{C(\partial\Omega\times[0,T])},\ M = \|F\|_{C(\Omega\times[0,T])},$$

and

$$f(t) = \int_0^t \kappa(\tau)\, d\tau,\ \ \int_0^t k(t-\tau)\kappa(\tau)\, d\tau \quad \equiv 1,\ \ t > 0.$$

Uniqueness of the solution

The initial-boundary-value problem for the general diffusion equation equation possesses at most one solution.

This solution continuously depends on the problem data in the sense that if u and $\tilde{u}$ solutions to the problems with the sources functions F and $\tilde{F}$ and the initial and boundary conditions u_0 and $\tilde{u}_0$ and v and $\tilde{v}$, respectively, and

$$\|F - \tilde{F}\|_{C(\bar{\Omega}\times[0,T])} \le \epsilon,$$

$$\|u_0 - \tilde{u}_0\|_{C(\bar{\Omega})} \le \epsilon_0, \ \|v - \tilde{v}\|_{C(\partial\Omega\times[0,T])} \le \epsilon_1,$$

then the following norm estimate holds true:

$$\|u - \tilde{u}\|_{C(\bar{\Omega}\times[0,T])} \le \max\{\epsilon_0, \epsilon_1\} + \epsilon\, f(T)$$

with

$$f(t) = \int_0^t \kappa(\tau)\, d\tau, \ \int_0^t k(t-\tau)\kappa(\tau)\, d\tau \quad \equiv 1, \ t > 0.$$

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Single-term time-fractional diffusion equation

Initial-boundary-value problem:

$$\partial_t^\alpha u(x,t) = \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u(x,t)) + c(x)u(x,t) + F(x,t),\ x \in \Omega \subset \mathbb{R}^n,\ t > 0$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0,$$

$$u(x,0) = a(x), \qquad x \in \Omega$$

with $0 < \alpha < 1$ and in a bounded domain Ω with a smooth boundary $\partial\Omega$.
In what follows, we always suppose that the spatial differential operator is a uniformly elliptic one.

Weak solution in the fractional Sobolev space

For $u \in C^1[0, T]$, the Caputo fractional derivative is defined by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_s u(x,s) ds, \quad x \in \Omega.$$

Recently, the Caputo fractional derivative ∂_t^α was extended to an operator defined on the closure $H_\alpha(0, T)$ of ${}_0C^1[0, T] := \{u \in C^1[0, T];\ u(0) = 0\}$ in the fractional Sobolev space $H^\alpha(\Omega)$.

In what follows, we interpret $\partial_t^\alpha u$ as this extension with the domain $H_\alpha(0, T)$.

Thus we interpret the problem under consideration as the fractional diffusion equation subject to the inclusions

$$\begin{cases} u(\cdot, t) \in H_0^1(\Omega), & t > 0, \\ u(x, \cdot) - a(x) \in H_\alpha(0, T), & x \in \Omega. \end{cases}$$

Results regarding the maximum principle

Luchko: Maximum principle for the strong solution under the assumption

$$c(x) \leq 0, \quad x \in \overline{\Omega}.$$

Luchko and Yamamoto: Maximum principle for the weak solution in the case $c \in C(\overline{\Omega})$ without the non-negativity condition $c(x) \leq 0,\ x \in \overline{\Omega}$.

Consequences from the maximum principle

Let us now denote the solution to the initial-boundary value problem for the fractional diffusion equation by $u_{a,F}$.
1) Non-negativity property:

Let $a \in L^2(\Omega)$ and $F \in L^2(\Omega \times (0,T))$. If $F(x,t) \geq 0$ a.e. (almost everywhere) in $\Omega \times (0,T)$ and $a(x) \geq 0$ a.e. in Ω, then $u_{a,F}(x,t) \geq 0$ a.e. in $\Omega \times (0,T)$.

2) Comparison property:

Let $a_1, a_2 \in L^2(\Omega)$ and $F_1, F_2 \in L^2(\Omega \times (0,T))$ satisfy the inequalities $a_1(x) \geq a_2(x)$ a.e. in Ω and $F_1(x,t) \geq F_2(x,t)$ a.e. in $\Omega \times (0,T)$, respectively. Then $u_{a_1,F_1}(x,t) \geq u_{a_2,F_2}(x,t)$ a.e. in $\Omega \times (0,T)$.

Comparison property regarding the coefficient $c = c(x)$

Let us now fix a source function $F = F(x,t) \geq 0$ and an initial condition $a = a(x) \geq 0$ and denote by $u_c = u_c(x,t)$ the weak solution to the time-fractional diffusion equation with the coefficient $c = c(x)$.

Then the following comparison property is valid:

Let $c_1, c_2 \in C(\overline{\Omega})$ satisfy the inequality $c_1(x) \geq c_2(x)$ in Ω. Then $u_{c_1}(x,t) \geq u_{c_2}(x,t)$ in $\Omega \times (0,T)$.

Positivity property

Conditions:
1) the initial condition $a \in L^2(\Omega)$, $a \geq 0$, $a \not\equiv 0$ a.e. in Ω,
2) the weak solution u belongs to $C((0,T]; C(\overline{\Omega}))$,
3) the source function is identically equal to zero, i.e.,
$F(x,t) \equiv 0, \ x \in \Omega, \ t > 0$.

Then the weak solution u is strictly positive:

$$u(x,t) > 0, \quad x \in \Omega, \, 0 < t \leq T.$$

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Abstract time- and space-fractional diffusion equation

Let X be a Hilbert space over $\boldsymbol{R}$ with the scalar product $(\cdot, \cdot)$. For $0 < \alpha, \beta < 1$, we consider the following evolution equation in the Hilbert space X:

$$D_t^\alpha u(t) = -(-A)^\beta u \quad \text{in } X,\ t > 0$$

along with the initial condition

$$u(0) = a \in X.$$

Assumptions: the operator A is self-adjoint, has compact resolvent, and $(-\infty, 0] \subset \rho(-A)$, $\rho(-A)$ being the resolvent of $-A$.

We note that $u(\cdot, t) := u(t) \in D((-A)^\beta)$ for $t > 0$ and so a boundary condition is incorporated into the equation.

Non-negativity property

For $0 < \alpha, \beta < 1$, let us denote a solution to the abstract time- and space-fractional diffusion equation by $u_{\alpha,\beta}(t)$.

If $a \geq 0$ in Ω, then $u_{\alpha,\beta}(\cdot, t) \geq 0$ in Ω for $t \geq 0$.

Sketch of the proof

The main idea is first to prove non-negativity of $u_{\alpha,\beta}$ in the case $\alpha = 1$ (space-fractional equation) and then to extend this result for the general case.

We start with the following important result:

$u_{1,\beta}(\cdot, t) \geq 0$ in Ω for $t \geq 0$ if $a \geq 0$ in Ω.

Sketch of the proof

Ingredients for the proof:

1) Integral representation

$$((-A)^\beta + 1)^{-1} a = \frac{\sin \pi\beta}{\pi} \int_0^\infty \frac{\mu^\beta(-A+\mu)^{-1} a}{\mu^{2\beta} + 2\mu^\beta \cos \pi\beta + 1} d\mu, \quad a \in X$$

2) Maximum principle for A => $(-A+\mu)^{-1} a \geq 0$ for $\mu \geq 0$ and $a \geq 0$ in Ω.

3)$\mu^{2\beta} + 2\mu^\beta \cos \pi\beta + 1 > 0$ for $\mu \geq 0$ and $0 < \beta < 1$.

Hence

$$(1 + (-A)^\beta)^{-1} a \geq 0 \quad \text{if} \;\; a(x) \geq 0, \; x \in \Omega.$$

Then

$$u_{1,\beta}(\cdot, t) = e^{-(-A)^\beta t} a = \lim_{\ell \to \infty} \left(1 + \frac{t}{\ell}(-A)^\beta\right)^{-\ell} a \geq 0.$$

Sketch of the proof

Let $0 < \alpha, \beta < 1$. Then

$$u_{\alpha,\beta}(x,t) = \int_0^\infty \Phi_\alpha(\eta) u_{1,\beta}(x, t^\alpha \eta) d\eta, \quad x \in \Omega,\ t > 0$$

with

$$\Phi_\alpha(\eta) = \sum_{\ell=0}^{\infty} \frac{(-\eta)^\ell}{\ell! \Gamma(-\alpha\ell + 1 - \alpha)}$$

being a particular case of the Wright function (also known as the Mainardi function).

Because $u_{1,\beta}(x,t) \geq 0$ for $x \in \Omega$ and $t \geq 0$ for $a \geq 0$ and

$$\Phi_\alpha(\eta) \geq 0, \quad \eta > 0$$

we get the inequality

$$u_{\alpha,\beta}(x,t) \geq 0,\ x \in \Omega,\ t \geq 0.$$

Some of other results

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Thank you very much for your attention!

$$(\mathbb{D}^C_{(k)} f)(t) = \int_0^t k(t-\tau) f'(\tau)\, d\tau$$

$$D_t^\alpha u(t) = -(-A)^\beta u$$

Questions and comments are welcome!