Müntz Spectral Methods with Applications to Some Singular Problems

Chuanju Xu 许传矩

School of Mathematical Sciences, Xiamen University

Collaborators: Dianming Hou (Xiamen U)

Brown U June 20, 2018

Summary

1. Motivation
   - Integro-differential equations
   - Fractional differential equations
   - Related works
2. Generalized fractional Jacobi polynomials
   - Preliminary
   - Fundamental properties of GFJPs
   - Projection, interpolation, and related error estimates
3. Müntz spectral method for some singular problems
   - Fractional integro-differential equations
   - Fractional elliptic equations
   - Time fractional diffusion equations

Motivation

We aim at constructing efficient numerical methods for a class of equations having singular solutions:

$$\begin{cases} u_t = a_1 u(t) + a_2 I_t^{\mu} u(t) + f(t), \quad t \in I, \mu \geq 0, \\ u(0) = 0. \end{cases}$$

$$\begin{cases} bu(x) - D_x^{\rho} u(x) = f(x), \quad x \in I, 1 < \rho < 2, \\ u(0) = 0, u_x(0) = u_1. \end{cases}$$

$$\begin{cases} D_t^{\alpha} u(x,t) - \partial_x^2 u(x,t) = f(x,t), \;\; I \times \Lambda, \;\; 0 < \alpha < 1, \\ u(x,0) = u_0, \\ u(x,t)|_{\partial\Lambda} = 0. \end{cases}$$

where $I = [0,1]$, a_1 and a_2 are real coefficients, and the operators I_t^{μ}, D_x^{ρ} denote the fractional integral and derivative.

Volterra integral equation

$$u(x) + \int_0^x (x-s)^{-\mu} K(x,s) u(s) = g(x),\ x \in \Lambda := (0,1), 0 < \mu < 1,$$

where $K(x,s)$ is a kernel function.
It has been well known [Brunner 2004] that: if $g \in C^m(\bar{\Lambda})$ and $K \in C^m(\bar{\Lambda} \times \bar{\Lambda})$ with $K(s,s) \neq 0$ in $\bar{\Lambda}$, then the solution can be expressed as

$$u(x) = \sum_{(j,k)\in G} \gamma_{j,k} x^{j+k(1-\mu)} + u_r(x),$$

where

$$G := \{(j,k) : j,k \text{ are non-negative integers s.t. } j + k(1-\mu) < m\},$$

$\gamma_{j,k}$ are constants, and $u_r(\cdot) \in C^m(\bar{\Lambda})$.

TFDE

$$\begin{cases} {}^R\partial_t^\alpha u - \partial_x^2 u = f \quad t \in I, x \in \Lambda, \alpha \in (0,1), \\ u(-1,t) = u(1,t) = 0 \quad t \in I. \end{cases}$$

or

$$\begin{cases} {}^C\partial_t^\alpha u - \partial_x^2 u = f \quad t \in I, x \in \Lambda, \alpha \in (0,1), \\ u(-1,t) = u(1,t) = 0 \quad t \in I, \\ u(x,0) = u_0(x) \quad x \in \Lambda. \end{cases}$$

Solution singularity

Solution representation in term of Mittag-Leffler function:

$$u(x,t)=\sum_{i=1}^{\infty}\Big[\int_0^t (f(\cdot,\tau),\psi_i)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-\tau)^{\alpha})d\tau\Big]\psi_i(x)$$
$$=t^{\alpha}\sum_{i=1}^{\infty}\Big[\int_0^1 (f(\cdot,\tau t),\psi_i)(1-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^{\alpha}(1-\tau)^{\alpha})d\tau\Big]\psi_i(x),$$

where

$$-\partial_x^2\psi_i(x)=\lambda_i\psi_i(x),\quad \psi_i(\pm 1)=0.$$

Even the forcing function f is smooth, the solution u may exhibit singularity with the leading order t^{α} at the starting point $t=0$ like the one for the Volterra integral equations.

The main difficulties:

- the operators I_t^{μ} and D_x^{ρ} are non-local;

- the solutions are usually singular near the boundary or at the starting time.

Spectral methods

Weak form of the TFDE

$$\begin{cases} {}^R\partial_t^\alpha u(x,t) - \triangle u(x,t) = f(x,t),\ t \in I := (0,T), x \in \Lambda := (-1,1), \\ u(-1,t) = u(1,t) = 0,\ t \in I. \end{cases}$$

Weak form: find $u \in B^{\frac{\alpha}{2}}(Q) := H^s(\Lambda, L^2(I)) \cap L^2(\Lambda, H_0^1(I))$, such that

$$\mathcal{A}(u,v) + \mathcal{B}(u,v) = (f,v), \quad \forall v \in B^{\frac{\alpha}{2}}(Q), \tag{1}$$

where $Q = \Lambda \times I$,

$$\mathcal{A}(u,v) := ({}_0\partial_t^{\frac{\alpha}{2}} u,\ {}_t\partial_T^{\frac{\alpha}{2}} v)_Q, \quad \mathcal{B}(u,v) := (\partial_x u, \partial_x v)_Q.$$

Spectral approximation

Let $L := (M, N)$, the space-time Galerkin spectral method reads: Find $u_L \in \mathcal{P}_M^0(\Lambda) \otimes \mathcal{P}_N(I)$, such that

$$\mathcal{A}(u_L, v_L) + \mathcal{B}(u_L, v_L) = \mathcal{F}(v_L), \quad \forall v_L \in \mathcal{P}_M^0(\Lambda) \otimes \mathcal{P}_N(I).$$

Theorem (*Li & Xu*, 2009)

If $u \in L^2(I, H^\sigma(\Lambda)) \cap H^\gamma(I, H_0^1(\Lambda)), \gamma > 1,\ \sigma \geqslant 1,$ *then*

$$\sqrt{\cos\frac{\pi\alpha}{2}}\|\partial_t^{\frac{\alpha}{2}}(u - u_L)\|_{0,Q} + \|\partial_x(u - u_L)\|_{0,Q}$$
$$\lesssim N^{\frac{\alpha}{2}-\gamma}\|u\|_{0,\gamma} + N^{\frac{\alpha}{2}-\gamma}M^{-\sigma}\|u\|_{\sigma,\gamma}$$
$$+M^{-\sigma}\|u\|_{\sigma,\frac{\alpha}{2}} + M^{1-\sigma}\|u\|_{\sigma,0} + N^{-\gamma}\|u\|_{1,\gamma}.$$

Other related works

- Polynomial spline collocation method for IDEs:

  [Brunner 1986], [Tang 1993], [Brunner et al. 2001], [Rawashdeh et al. 2004], [Tarang 2004].

- Spectral method for Volterra integral equations(VIEs) with nonsmooth solution:

  [Chen and Tang 2010], [Li, Tang, and Xu 2015], [Stynes and Huang 2016].

- Non-polynomial basis for FDEs:

  [Zayernouri and Karniadakis 2013, 2014, …], [Chen, Shen, and Wang 2016].

- Mapped Jacobi and Müntz-Legendre functions for Elliptic equations: [Shen and Wang 2016].

Müntz polynomials

The well-known Weierstrass theorem states:
every continuous function on a compact interval can be uniformly approximated by algebraic polynomials.

This result was generalized by Bernstein 1912, and proved by Müntz (theorem) 1914:

the Müntz polynomials of the form $\sum_{k=0}^{n} a_k x^{\lambda_k}$ *with real coefficients, i.e., span*$\{x^{\lambda_k}, k = 0, 1, \dots\}$, *are dense in* $C^0[0,1]$ *if and only if* $\sum_{k=1}^{\infty} \lambda_k^{-1} = +\infty$, *where* $\{\lambda_0, \lambda_1, \lambda_2, \dots\}$ *is a sequence of distinct positive numbers such that* $0 = \lambda_0 < \lambda_1 < \ldots \to \infty$.

Extension to $L^2(0,1)$ **by Szász 1916**.

Generalized fractional Jacobi polynomials (GFJPs)

We will make new use of Müntz polynomial spaces defined by

$$P_N^\lambda(I) = span\{1, x^\lambda, x^{2\lambda}, \cdots, x^{N\lambda}\}, \quad 0 < \lambda \le 1.$$

Generalized fractional Jacobi polynomials

$$J_{n+\ell}^{\alpha,\beta,\lambda}(x) = \begin{cases} J_n^{\alpha,\beta}(2x^\lambda - 1), & \alpha, \beta > -1, \\ \frac{n+\alpha+1}{n+1} x^\lambda J_n^{\alpha,1}(2x^\lambda - 1), & \alpha > -1, \beta = -1, \\ \frac{n+\beta+1}{n+1}(1 - x^\lambda) J_n^{1,\beta}(2x^\lambda - 1), & \alpha = -1, \beta > -1, \\ -(1 - x^\lambda) x^\lambda J_n^{1,1}(2x^\lambda - 1), & \alpha = \beta = -1, \end{cases}$$

where $J_n^{\alpha,\beta}(x)$ denote the classical Jacobi polynomials, and

$$\ell = \begin{cases} 0, & \alpha, \beta > -1, \\ 1, & \alpha = -1, \beta > -1 \text{ or } \alpha > -1, \beta = -1, \\ 2, & \alpha = \beta = -1. \end{cases}$$

Some fundamental properties of GFJPs

Lemma

The generalized fractional Jacobi polynomials $J_n^{\alpha,\beta,\lambda}(x)$ *are mutually orthogonal with respect to the weight function* $\omega^{\alpha,\beta,\lambda}(x) = \lambda(1-x^\lambda)^\alpha x^{(\beta+1)\lambda-1}, \alpha, \beta \geq -1, 0 < \lambda \leq 1$, *i.e.,*

$$\int_0^1 \omega^{\alpha,\beta,\lambda}(x) J_n^{\alpha,\beta,\lambda}(x) J_m^{\alpha,\beta,\lambda}(x) dx = \gamma_n^{\alpha,\beta} \delta_{m,n},$$

where

$$\gamma_n^{\alpha,\beta} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}.$$

Furthermore,

$$\partial_x J_n^{\alpha,\beta,\lambda}(x) = (n+\alpha+\beta+1) J_{n-1}^{\alpha+1,\beta+1,\lambda}(x).$$

Sturm-Liouville problem:

Lemma

The generalized fractional Jacobi polynomials $\{J_n^{\alpha,\beta,\lambda}\}_{n=0}^{\infty}$ *with* $\alpha, \beta \geq -1$ *satisfy the following Sturm-Liouville problem:*

$$(\omega^{\alpha,\beta,\lambda}(x))^{-1}\partial_x\{\lambda^{-1}(1-x^{\lambda})^{\alpha+1}x^{\beta\lambda+1}\partial_x J_n^{\alpha,\beta,\lambda}(x)\}$$

$$= -\sigma_n^{\alpha,\beta} J_n^{\alpha,\beta,\lambda}(x),$$

where $\sigma_n^{\alpha,\beta} = n(n+\alpha+\beta+1)$.

New differentiation operators

Differentiation operators:

$$D_\lambda^0 := I_d, \quad D_\lambda := \frac{d}{dx^\lambda} := \frac{d}{\lambda x^{\lambda-1} dx}, \quad D_\lambda^2 := D_\lambda D_\lambda, \cdots,$$

$$D_\lambda^k := \overbrace{D_\lambda D_\lambda \cdots D_\lambda}^{k}, \; k = 0, 1, \cdots.$$

Define:

$$^{+}D_\lambda^1 v(x) := \lim_{\Delta x \to 0^+} \frac{v(x+\Delta x) - v(x)}{(x+\Delta x)^\lambda - x^\lambda},$$

$$^{-}D_\lambda^1 v(x) := \lim_{\Delta x \to 0^-} \frac{v(x+\Delta x) - v(x)}{(x+\Delta x)^\lambda - x^\lambda}.$$

Then $D_\lambda^1 v(x)$ exists if and only if $^{+}D_\lambda^1 v(x) = {}^{-}D_\lambda^1 v(x)$, and $D_\lambda^1 v(x) = {}^{+}D_\lambda^1 v(x) = {}^{-}D_\lambda^1 v(x)$.

Connection with local fractional derivatives

Remark:

- This derivative was called Hausdorff derivative, introduced in [Chen 06] for fractal time-space fabric, and studied in [Weberszpil et al. 2015], [Chen et al. 2017], [Chen 2017], ...

- It is also closely related to the local fractional derivatives used in Fractals; see [Li et al. 2013], [Lutton & Tricot (eds), Fractals. Springer, 1999], [Chen et al. 2010], ...

Using this new derivative, and set the weight function:

$$\widehat{\omega}^{\alpha,\beta,\lambda}(x) := (1-x^{\lambda})^{\alpha}x^{\beta\lambda} = \lambda^{-1}x^{1-\lambda}\omega^{\alpha,\beta,\lambda}(x).$$

Then the fractional Jacobi polynomials $\{J_n^{\alpha,\beta,\lambda}\}_{n=0}^{\infty}$ satisfy the following singular Sturm-Liouville problem:

$$\mathcal{L}_{\lambda}^{\alpha,\beta} J_n^{\alpha,\beta,\lambda}(x) = \sigma_n^{\alpha,\beta} J_n^{\alpha,\beta,\lambda}(x),$$

where $\sigma_n^{\alpha,\beta} = n(n+\alpha+\beta+1)$, the singular Sturm-Liouville operator $\mathcal{L}_{\lambda}^{\alpha,\beta}$ is defined by

$$\mathcal{L}_{\lambda}^{\alpha,\beta} v(x) = -(\widehat{\omega}^{\alpha,\beta,\lambda}(x))^{-1} D_{\lambda}^{1}\{(1-x^{\lambda})^{\alpha+1}x^{(\beta+1)\lambda} D_{\lambda}^{1} v(x)\}.$$

Lemma

The new defined k-th order derivatives of the fractional Jacobi polynomials are orthogonal with respect to the weight $\omega^{\alpha+k,\beta+k,\lambda}(x)$, i.e.,

$$\int_0^1 \omega^{\alpha+k,\beta+k,\lambda}(x) D_\lambda^k J_n^{\alpha,\beta,\lambda}(x) D_\lambda^k J_m^{\alpha,\beta,\lambda}(x)dx = \widehat{h}_{n,k}^{\alpha,\beta}\delta_{m,n},$$

where

$$\widehat{h}_{n,k}^{\alpha,\beta} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+k+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(n-k)!\Gamma^2(n+\alpha+\beta+1)}.$$

Moreover, we have

$$D_\lambda^k J_n^{\alpha,\beta,\lambda}(x) = \widehat{d}_{n,k}^{\alpha,\beta} J_{n-k}^{\alpha+k,\beta+k,\lambda}(x),$$

where

$L^2_{\omega^{\alpha,\beta,\lambda}}(I)$-orthogonal projector with $\alpha, \beta > -1$

Let $\pi_{N,\omega^{\alpha,\beta,\lambda}} : L^2_{\omega^{\alpha,\beta,\lambda}}(I) \to P_N^\lambda(I)$ be the $L^2_{\omega^{\alpha,\beta,\lambda}}$-orthogonal projector defined by: for all $v \in L^2_{\omega^{\alpha,\beta,\lambda}}(I)$, $\pi_{N,\omega^{\alpha,\beta,\lambda}} v \in P_N^\lambda(I)$ such that

$$(v - \pi_{N,\omega^{\alpha,\beta,\lambda}} v, v_N)_{\omega^{\alpha,\beta,\lambda}} = 0, \quad \forall v_N \in P_N^\lambda(I).$$

Equivalently, $\pi_{N,\omega^{\alpha,\beta,\lambda}}$ can be characterized by:

$$\pi_{N,\omega^{\alpha,\beta,\lambda}} v(x) = \sum_{n=0}^{N} \hat{v}_n^{\alpha,\beta} J_n^{\alpha,\beta,\lambda}(x),$$

where $J_n^{\alpha,\beta,\lambda}(x)$ are the fractional Jacobi polynomials, and

$$\hat{v}_n^{\alpha,\beta} = \frac{(v, J_n^{\alpha,\beta,\lambda})_{\omega^{\alpha,\beta,\lambda}}}{\|J_n^{\alpha,\beta,\lambda}\|^2_{0,\omega^{\alpha,\beta,\lambda}}}.$$

Some spaces

To measure the projection error, we need non-uniform fractional Jacobi-weighted Sobolev spaces:

$$B^m_{\omega^{\alpha,\beta,\lambda}}(I) := \left\{ v : D^k_\lambda v \in L^2_{\omega^{\alpha+k,\beta+k,\lambda}}(I), 0 \le k \le m \right\},\ m = 0, 1, 2, \cdots,$$

equipped with the inner product, norm, and semi-norm:

$$(u, v)_{B^m_{\omega^{\alpha,\beta,\lambda}}} = \sum_{k=0}^{m} (D^k_\lambda u, D^k_\lambda)_{\omega^{\alpha+k,\beta+k,\lambda}},$$

$$\|v\|_{m,\omega^{\alpha,\beta,\lambda}} = (v, v)^{1/2}_{B^m_{\omega^{\alpha,\beta,\lambda}}}, \quad |v|_{m,\omega^{\alpha,\beta,\lambda}} = \|D^m_\lambda v\|_{0,\omega^{\alpha+m,\beta+m,\lambda}}.$$

The special case $\lambda = 1$ gives the classical non-uniform Jacobi-weighted Sobolev spaces:

$$B^m_{\omega^{\alpha,\beta,1}}(I) := \left\{ v : \partial^k_x v \in L^2_{\omega^{\alpha+k,\beta+k,1}}(I), 0 \le k \le m \right\}.$$

Lemma

The orthogonal projector $\pi_{N,\omega^{\alpha,\beta,\lambda}}$ *admits the following error estimate: for any* $v(x^{\frac{1}{\lambda}}) \in B^{m,1}_{\alpha,\beta}(I)$, *and* $0 \le l \le m \le N+1$,

$$\|D^l_\lambda(v - \pi_{N,\omega^{\alpha,\beta,\lambda}}v)\|_{0,\omega^{\alpha+l,\beta+l,\lambda}}$$

$$\le c\sqrt{\frac{(N-m+1)!}{(N-l+1)!}}(N+m)^{(l-m)/2}\|\partial^m_x\{v(x^{\frac{1}{\lambda}})\}\|_{0,\omega^{\alpha+m,\beta+m,1}}.$$

For a fixed m, *the above estimate can be simplified as*

$$\|D^l_\lambda(v - \pi_{N,\omega^{\alpha,\beta,\lambda}}v)\|_{0,\omega^{\alpha+l,\beta+l,\lambda}} \le cN^{l-m}\|\partial^m_x\{v(x^{\frac{1}{\lambda}})\}\|_{0,\omega^{\alpha+m,\beta+m,1}}.$$

In particular, for $l = 0, 1$, *we have*

$$\|v - \pi_{N,\omega^{\alpha,\beta,\lambda}}v\|_{0,\omega^{\alpha,\beta,\lambda}} \le cN^{-m}\|\partial^m_x\{v(x^{\frac{1}{\lambda}})\}\|_{0,\omega^{\alpha+m,\beta+m,1}}$$

$$\|\partial_x(v - \pi_{N,\omega^{\alpha,\beta,\lambda}}v)\|_{0,\tilde{\omega}^{\alpha,\beta,\lambda}} \le cN^{1-m}\|\partial^m_x\{v(x^{\frac{1}{\lambda}})\}\|_{0,\omega^{\alpha+m,\beta+m,1}},$$

where $\tilde{\omega}^{\alpha,\beta,\lambda} = \lambda^{-1}(1-x^\lambda)^{\alpha+1}x^{\beta\lambda+1}$.

$L^2_{\omega^{\alpha,\beta,\lambda}}-$projector with $\alpha, \beta \geq -1$.

Define the fractional polynomial spaces for $\alpha, \beta \geq -1$:

$$S_{N,\lambda}^{\alpha,\beta} := \text{span}\{J_{i+\ell}^{\alpha,\beta,\lambda}(x), i = 0, 1, 2, \cdots, N\}$$

$L^2_{\omega^{\alpha,\beta,\lambda}}-$projector $\pi_{N,\omega^{\alpha,\beta,\lambda}}$: $L^2_{\omega^{\alpha,\beta,\lambda}}(I) \to S_{N,\lambda}^{\alpha,\beta}$, $\forall v \in L^2_{\omega^{\alpha,\beta,\lambda}}(I)$,

$$(v - \pi_{N,\omega^{\alpha,\beta,\lambda}} v,\ v_N)_{\omega^{\alpha,\beta,\lambda}} = 0, \quad \forall v_N \in S_{N,\lambda}^{\alpha,\beta}.$$

For special case $\beta = -1$, we define the dual fractional polynomial space of $S_{N,\lambda}^{\alpha,-1}$ as follows:

$$V_{N,\lambda}^{-\alpha-1,0} := \text{span}\Big\{(1 - x^{\lambda})^{\alpha+1} J_j^{\alpha+1,0,\lambda}(x), j = 0, 1, 2, \cdots, N\Big\}.$$

Approximation results

Theorem

For any $v(x)$ such that $v(x^{\frac{1}{\lambda}}) \in B^m_{\omega^{\alpha,\beta,1}}(I), m \geq 1$, its orthogonal projection $\pi_{N,\omega^{\alpha,\beta,\lambda}} v$ admits the following optimal error estimates:

$$\|v - \pi_{N,\omega^{\alpha,\beta,\lambda}} v\|_{0,\omega^{\alpha,\beta,\lambda}} \leq cN^{-m} \|\partial_x^m v(x^{\frac{1}{\lambda}})\|_{0,\omega^{m+\alpha,m+\beta,1}},$$

$$\|\partial_x (v - \pi_{N,\omega^{\alpha,\beta,\lambda}} v)\|_{0,\hat{\omega}^{\alpha,\beta,\lambda}} \leq cN^{1-m} \|\partial_x^m v(x^{\frac{1}{\lambda}})\|_{0,\omega^{m+\alpha,m+\beta,1}},$$

where $\hat{\omega}^{\alpha,\beta,\lambda} = \lambda^{-1}(1 - x^\lambda)^{\alpha+1} x^{\beta\lambda+1}$.

Remark

It is shown that even if $v(x)$ is singular its projection $\pi_{N,\omega^{\alpha,\beta,\lambda}} v$ can be a very good approximation to $v(x)$ if λ is properly chosen such that $v(x^{1/\lambda})$ is smooth or $v(x^{1/\lambda}) \in B^m_{\omega^{\alpha,\beta,1}}(I)$ for large m.

$I_{N,\lambda}^{\alpha,\beta}$-interpolation on fractional Jacobi-Gauss-type points

Let $h_{j,\lambda}^{\alpha,\beta}(x)$ be the generalized Lagrange basis function:

$$h_{j,\lambda}^{\alpha,\beta}(x) = \prod_{i=0,i\neq j}^{N} \frac{x^\lambda - x_i^\lambda}{x_j^\lambda - x_i^\lambda}, \quad 0 \le j \le N,$$

where $x_0 < x_1 < \cdots < x_{N-1} < x_N$ are zeros in I of $J_{N+1}^{\alpha,\beta,\lambda}(x)$. It is clear that the functions $h_{j,\lambda}^{\alpha,\beta}(x)$ satisfy

$$h_{j,\lambda}^{\alpha,\beta}(x_i) = \delta_{ij}.$$

Let $z(x) = x^\lambda$. Then $z_i := z(x_i) = x_i^\lambda, \ \ 0 \le i \le N$ are zeros of $J_{N+1}^{\alpha,\beta,1}(x)$, and

$$h_{j,\lambda}^{\alpha,\beta}(x) = h_{j,1}^{\alpha,\beta}(z) := \prod_{i=0,i\neq j}^{N} \frac{z - z_i}{z_j - z_i}, \quad 0 \le j \le N.$$

We define the interpolation operator $I_{N,\lambda}^{\alpha,\beta}$ by

$$I_{N,\lambda}^{\alpha,\beta} v(x) = \sum_{j=0}^{N} v(x_j) h_{j,\lambda}^{\alpha,\beta}(x).$$

Lemma

For any $v(x^{\frac{1}{\lambda}}) \in B_{\alpha,\beta}^{1,1}$*, we have*

$$\|I_{N,\lambda}^{\alpha,\beta} v\|_{0,\omega^{\alpha,\beta,\lambda}} \leq c\left(\|v\|_{0,\omega^{\alpha,\beta,\lambda}} + N^{-1}\|D_{\lambda}^{1} v\|_{0,\omega^{\alpha+1,\beta+1,\lambda}}\right).$$

For any $v(x^{1/\lambda}) \in B_{\alpha,\beta}^{m,1}(I)$, $m \geq 1$, *and* $0 \leq l \leq m \leq N+1$, *it holds*

$$\|D_{\lambda}^{l}\left(v - I_{N,\lambda}^{\alpha,\beta} v\right)\|_{0,\omega^{\alpha+l,\beta+l,\lambda}}$$
$$\leq c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{l-(m+1)/2}\|\partial_x^m\left\{v(x^{\frac{1}{\lambda}})\right\}\|_{0,\omega^{\alpha+m,\beta+m,1}}.$$

For given m*,*

$$\|D_{\lambda}^{l}\left(v - I_{N,\lambda}^{\alpha,\beta} v\right)\|_{0,\omega^{\alpha+l,\beta+l,\lambda}} \leq cN^{l-m}\|\partial_x^m\left\{v(x^{\frac{1}{\lambda}})\right\}\|_{0,\omega^{\alpha+m,\beta+m,1}}.$$

Interpolation error in $L^\infty-$norm

Lemma

If $-1 < \alpha, \beta \leq -\frac{1}{2}, v(x^{1/\lambda}) \in B^{m,1}_{\alpha,\beta}(I),\ m \geq 1.$ *Then*

$$\|v - I^{\alpha,\beta}_{N,\lambda} v\|_\infty \leq cN^{1/2-m} \|\partial_x^m v(x^{1/\lambda})\|_{0,\omega^{\alpha+m,\beta+m,1}}.$$

Petrov-Galerkin method for IDEs

Consider the Petrov-Galerkin based Müntz spectral method for IDEs: Find $u_N \in S_{N,\lambda}^{0,-1}(I)$, such that

$$(u_N', v_N) = a_1(u_N, v_N) + a_2({}_0I_t^{\mu} u_N, v_N) + (f, v_N), \forall v_N \in V_{N,\lambda}^{-1,0}(I).$$

Notice the facts

$$\omega^{1,\frac{1}{\lambda}-2,\lambda} v_N = \lambda t^{-\lambda}(1-t^{\lambda}) v_N \in V_{N,\lambda}^{-1,0}(I), \quad \forall v_N \in S_{N,\lambda}^{0,-1}(I).$$

We have the equivalent weighted Galerkin form: Find $u_N \in S_{N,\lambda}^{0,-1}(I)$, such that

$$\begin{aligned}(u_N', v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} &= a_1(u_N, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} + a_2({}_0I_t^{\mu} u_N, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}} \\ &\quad + (f, v_N)_{\omega^{1,\frac{1}{\lambda}-2,\lambda}}, \quad \forall v_N \in S_{N,\lambda}^{0,-1}(I).\end{aligned}$$

Theorem

If the coefficients a_1 *and* a_2 *satisfy*

$$a_1 \leq 0, |a_2| < \frac{\sqrt{2\mu e}\Gamma(\mu+1/2)}{2\Gamma(1/2)},$$

or

$$a_1 > 0, \frac{a_1}{e} + \frac{|a_2|\Gamma(1/2)}{\sqrt{2\mu e}\Gamma(\mu+1/2)} < \frac{1}{2}.$$

Then the Müntz spectral approximation problem admits a unique solution. Furthermore, if the exact solution $u(t)$ *such that* $u(t^{\frac{1}{\lambda}}) \in B^m_{\omega^{0,-1,1}}(I)$, *the following optimal error estimate holds:*

$$\|u - u_N\|_{0,\omega^{0,-1,\lambda}} \leq cN^{-m}\|\partial_t^m u(t^{\frac{1}{\lambda}})\|_{0,\omega^{m,m-1,1}}.$$

Müntz spectral method for fractional elliptic equations

$$\begin{cases} bu(x) - D_x^{\rho} u(x) = f(x), \quad x \in I, 1 < \rho < 2, \\ u(0) = 0, u_x(0) = u_1. \end{cases}$$

Applying Riemann-Liouville integral of order $\rho - 1$ to the both sides of the equation, and noticing that

$$I_x^{\rho-1} D_x^{\rho} u(x) = I_x^{\rho-1} I_x^{2-\rho} u_{xx} = I_x^1 u_{xx} = u_x - u_x(0) = u_x - u_1,$$

we get the following equivalent integro-differential equation:

$$\begin{cases} u_x = b\, I_x^{\rho-1} u(x) - I_x^{\rho-1} f(x) + c_0, \quad x \in I, 1 < \rho < 2, \\ u(0) = 0. \end{cases}$$

Therefore the Müntz spectral method constructed for IDEs can be directly applied.

TFDE

$$\begin{cases} D_t^\alpha u(x,t) - \partial_x^2 u(x,t) = f(x,t), \;\; I \times \Lambda, \; 0 < \alpha < 1, \\ u(x,0) = 0, \\ u(x,t)|_{\partial\Lambda} = 0. \end{cases}$$

Weak form: given f satisfying ${}_0I_t^{\mu/2} f(x,t) \in L^2(\Omega)$, find $u \in \mathcal{H}^{\mu/2}(\Omega) := {}_0H^{\mu/2}(I, L^2(\Lambda)) \cap L^2(I, H_0^1(\Lambda))$ such that

$$\mathcal{A}(u,v) = \mathcal{F}(v), \; \forall v \in \mathcal{H}^{\mu/2}(\Omega),$$

where the bilinear form $\mathcal{A}(\cdot,\cdot)$ is defined by

$$\mathcal{A}(u,v) := ({}_0D_t^{\mu/2} u, {}_tD_1^{\mu/2} v)_\Omega + (\partial_x u, \partial_x v)_\Omega,$$

and the functional $\mathcal{F}(\cdot)$ is given by

$$\mathcal{F}(v) := (f,v)_\Omega.$$

Theorem

For any $0 < \mu < 1$ *and* ${}_0I_t^{\mu/2} f \in L^2(\Omega)$, *the problem is well-posed. Furthermore, if* u *is the solution, then it holds*

$$\|u\|_{\mathcal{H}^{\mu/2}(\Omega)} \lesssim \|{}_0I_t^{\mu/2} f\|_{0,\Omega}.$$

Let

$$P_M^0(\Lambda) := P_M(\Lambda) \cap H_0^1(\Lambda),$$

$$S_{N,\lambda}^{\alpha,-1}(I) = \text{span}\{J_{i+1}^{\alpha,-1,\lambda}(x), i = 0, 1, 2, \cdots, N\}, \alpha > -1.$$

$L := (M, N)$,

$$S_L^\alpha(\Omega) := P_M^0(\Lambda) \otimes S_{N,\lambda}^{\alpha,-1}(I).$$

Müntz spectral Galerkin method: find $u_L \in S_L^\alpha(\Omega)$, such that

$$\mathcal{A}(u_L, v_L) = \mathcal{F}(v_L), \quad \forall v_L \in S_L^\alpha(\Omega).$$

Theorem

Let $0 < \mu < 1, -1 < \alpha \leq -\mu/2$. *Suppose* $u(x, t^{1/\lambda}) \in B^m_{\omega^{\alpha,-1,1}}(I, H^\sigma(\Lambda)) \cap B^m_{\omega^{\alpha,-1,1}}(I, H^1_0(\Lambda)), m \geq 1, \sigma \geq 1$. *Then the solution* u_L *of the Müntz spectral approximation satisfies:*

$$
\begin{aligned}
&\|u - u_L\|_{\mathcal{H}^{\mu/2}(\Omega)} \\
\lesssim\, & N^{\frac{1}{2}-m} \Big\| \|\partial_t^m v(\cdot, t^{1/\lambda})\|_{0,\Lambda} \Big\|_{0,\omega^{\alpha+m,m-1,1}} \\
& + N^{\frac{1}{2}-m} M^{-\sigma} \Big\| \|\partial_t^m v(\cdot, t^{1/\lambda})\|_{\sigma,\Lambda} \Big\|_{0,\omega^{\alpha+m,m-1,1}} \\
& + M^{-\sigma} \|v\|_{\sigma,s} + M^{1-\sigma} \|v\|_{\sigma,0} \\
& + N^{-m} \Big\| \|\partial_t^m v(\cdot, t^{1/\lambda})\|_{1,\Lambda} \Big\|_{0,\omega^{\alpha+m,m-1,1}}.
\end{aligned}
$$

Classical elliptic problems

$$\begin{cases} -\partial_x^2 u(x) = f(x), \ x \in I, \\ u(0) = u(1) = 0. \end{cases}$$

Weak form: For $f \in L^2_{\omega^{1,4/\lambda-3,\lambda}}(I)$, find $u \in B^1_{\omega^{-1,-1,\lambda}}(I)$, such that

$$\mathcal{A}(u,v) = \mathcal{F}(v), \quad \forall v \in B^1_{\omega^{-1,-1,\lambda}}(I),$$

where the bilinear form $\mathcal{A}(\cdot,\cdot)$ is defined by

$$\mathcal{A}(u,v) = \big(\partial_x u(x), \partial_x\{\omega^{0,2/\lambda-2,\lambda}(x)v(x)\}\big),$$

and the functional $\mathcal{F}(\cdot)$ is given by

$$\mathcal{F}(v) = (f(x), v(x))_{\omega^{0,2/\lambda-2,\lambda}}.$$

In order to prove the well-posedness of this problems, we need following Poincaré inequality:

For all $\lambda \in (0,1]$, Poincaré inequality in $B^1_{\omega^{-1,-1,\lambda}}(I)$ holds

$$\|v\|_{0,\omega^{-1,-1,\lambda}} \leq c\|\partial_x v\|_{0,\omega^{0,2/\lambda-2,\lambda}}, \quad \forall v \in B^1_{\omega^{-1,-1,\lambda}}(I).$$

Theorem

For all $f \in L^2_{\omega^{1,4/\lambda-3,\lambda}}(I)$, the discrete problem is well-posed. Furthermore, if u is the solution, it holds

$$\|u\|_{1,\omega^{-1,-1,\lambda}} \leq c\|f\|_{0,\omega^{1,4/\lambda-3,\lambda}}.$$

Müntz spectral method: Find $u_N \in B^1_{\omega^{-1,-1,\lambda}}(I) \cap S^{-1,-1}_{N,\lambda}(I)$, such that

$$\mathcal{A}(u_N,\ v_N) = \mathcal{F}(v_N), \quad \forall v_N \in B^1_{\omega^{-1,-1,\lambda}}(I) \cap S^{-1,-1}_{N,\lambda}(I).$$

Theorem

For all $f \in L^2_{\omega^{1,4/\lambda-3,\lambda}}(I)$, the Müntz spectral discrete problem admits a unique solution u_N, which satisfies

$$\|u_N\|_{1,\omega^{-1,-1,\lambda}} \le C\|f\|_{0,\omega^{1,4/\lambda-3,\lambda}}.$$

Furthermore, if $u(x^{1/\lambda}) \in B^m_{\omega^{-1,-1,1}}(I)$, then

$$\|u - u_N\|_{1,\omega^{-1,-1,\lambda}} \le cN^{1-m}\|\partial_x^m u(x^{1/\lambda})\|_{0,\omega^{m-1,m-1,1}}.$$

Fractional Jacobi Spectral-Collocation Method for VIEs

Volterra integral equation

$$u(x) = g(x) + \int_0^x (x-s)^{-\mu} K(x,s) u(s) ds, \quad 0 < \mu < 1, \;\; x \in I := [0,1].$$

Consider the fractional Jacobi spectral-collocation method as follows: find fractional polynomial $u_N^\lambda \in P_N^\lambda(I)$, such that

$$u_N^\lambda(x_i) = g(x_i) + \left(\mathcal{K} u_N^\lambda\right)(x_i), \quad 0 \le i \le N,$$

where the collocation points $\{x_i\}_{i=0}^N$ are roots of $J_{N+1}^{\alpha,\beta,\lambda}(x)$,

$$\left(\mathcal{K}\varphi\right)(x_i) = \int_0^{x_i} (x_i - s)^{-\mu} K(x_i, s) \varphi(s) ds.$$

Theorem

Let $u(x)$ *be the exact solution to the Volterra integral equation and* $u_N^\lambda(x)$ *is the numerical solution of the fractional Jacobi spectral-collocation problem. Assume* $0 < \mu < 1, -1 < \alpha, \beta \le -\frac{1}{2}$, $K(x,s) \in C^m(I,I)$ *and* $u(x^{\frac{1}{\lambda}}) \in B^{m,1}_{\alpha,\beta}(I), m \ge 1$. *Then we have*

$$\|u - u_N^\lambda\|_\infty \le cN^{\frac{1}{2}-m}(\|\partial_x^m u(x^{\frac{1}{\lambda}})\|_{0,\omega^{\alpha+m,\beta+m,1}} + N^{-\frac{1}{2}}\log N K^*\|u\|_\infty),$$

where K^* *is a constant only depending on* $K(\cdot,\cdot)$.

Numerical results: Example 1

- We start by considering the IDEs:

$$\begin{cases} u_t = u(t) + {}_0I_t^{\mu} u(t) + f(t), \quad t \in I, \mu \geq 0, \\ u(0) = 0, \end{cases}$$

with the source term $f(t) = 1/2t^{-1/2} - \Gamma(3/2)t - t^{1/2}$ and $\mu = 1/2$.
The exact solution: $u(t) = t^{1/2}$.

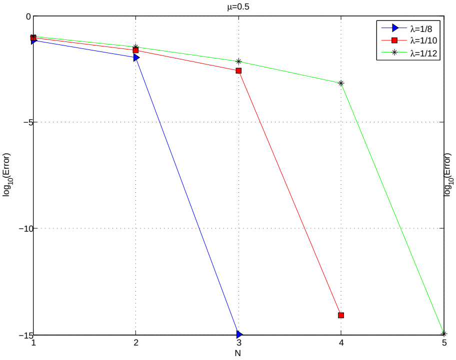


Figure:1 $L^2_{\omega^{0,-1,\lambda}}$ −NORM

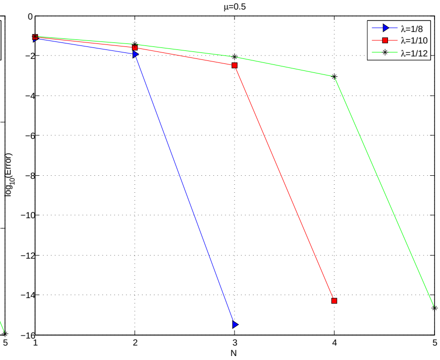


Figure:2 $B^1_{\omega^{0,-1,\lambda}}$ −NORM

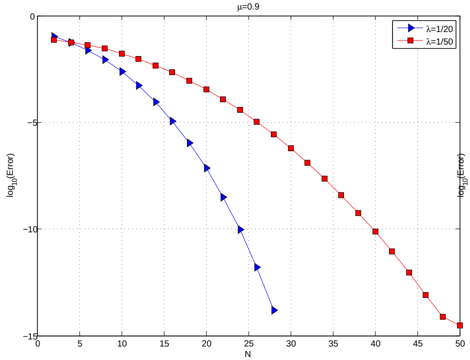
Example 2

- Consider IDEs with source term

$$f(t) = 1 - \frac{1}{\Gamma(2+\mu)} t^{1+\mu} - t + \sqrt{3} t^{\sqrt{3}-1} - \frac{\Gamma(\sqrt{3}+1)}{\Gamma(\sqrt{3}+1+\mu)} t^{\sqrt{3}+\mu} - t^{\sqrt{3}}$$

$\mu = 0.9$.

Exact solution: $u(t) = t + t^{\sqrt{3}}$.



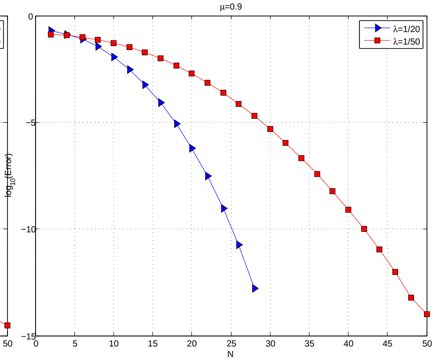


Figure:3 $L^2_{\omega^{0,-1,\lambda}}$ −NORM

Figure:4 $B^1_{\omega^{0,-1,\lambda}}$ −NORM

Example 3

- Consider an arbitrary smooth force function $f(t) = \sin(4\pi t)$.

$$u(t) = 2\pi t^2 + \gamma_{3,1} t^{3+\mu} + \sum_{j+k\mu > 3+\mu} \gamma_{j,k} t^{j+k\mu} + u_s(t).$$

where $\gamma_{j,k}$ are constants, and $u_s(\cdot) \in C^\infty(I)$.

It is seen that $u(t^{1/\lambda}) \in B^{2(3+\mu)/\lambda-\varepsilon}_{\omega^{0,-1,1}}(I)$ for any $\varepsilon > 0$.

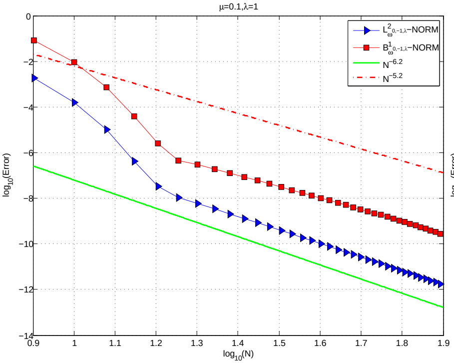


Figure: $\mu = 0.1, \lambda = 1.$

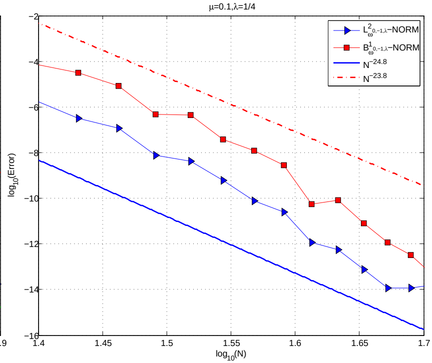


Figure:6 $\mu = 0.1, \lambda = 1/4.$

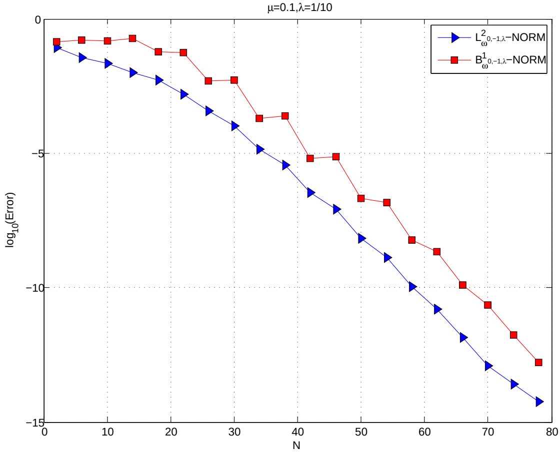
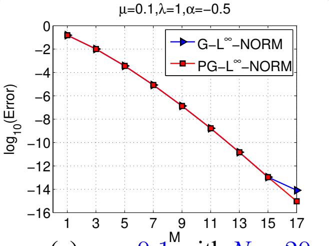


Figure:7 $\mu = 0.1, \lambda = 1/10.$

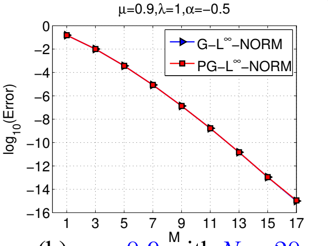
TFDE

- Consider TFDE for $\mu = 0.1, 0.9$. The fabricated exact solution is:

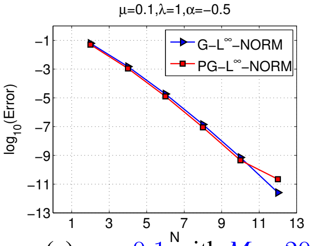
$$u(x,t) = \sin \pi t \sin \pi x.$$



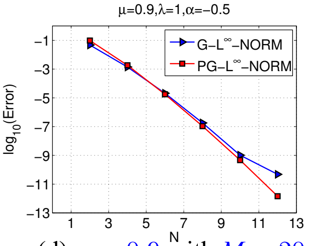
(a) $\mu = 0.1$ with $N = 20$



(b) $\mu = 0.9$ with $N = 20$



(c) $\mu = 0.1$ with $M = 20$



(d) $\mu = 0.9$ with $M = 20$

Figure: Error decays of the numerical solutions with respect to the polynomial degrees for the smooth exact solution.

TFDE

- Consider TFDE with smooth force function $f(x,t) = sin(\pi x) sin(\pi t)$, the exact solution is unknown.

  Serve the numerical solution calculated with $M = 40, N = 100$ as the “exact” solution.

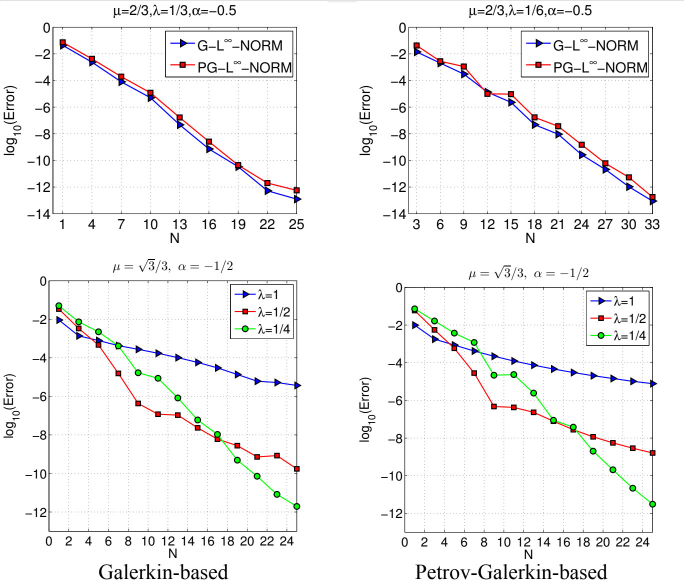


Figure: Errors versus N for different μ and λ.

Example 4

- Consider the elliptic problem

$$\begin{cases} -\partial_x^2 u(x) = f(x), \; x \in I, \\ u(0) = u(1) = 0, \end{cases}$$

with two source terms:
$(i)\ f(x) = \pi^2 \sin(\pi x)$
$(ii)\ f(x) = \frac{12}{169} x^{-14/13}$

Case (i): $u(x) = sin(\pi x)$

Case (ii): $u(x) = x^{12/13} - x$

Case (i) with smooth solution

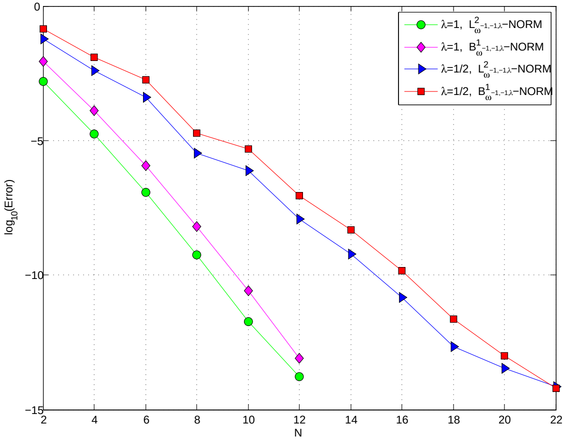


Figure:8 $u(x) = \sin(\pi x)$

Case (ii) with limited regular solution

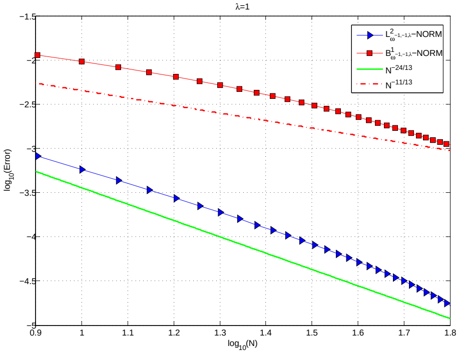


Figure:9 $u(x) = x^{12/13} - x$ with $\lambda = 1$.

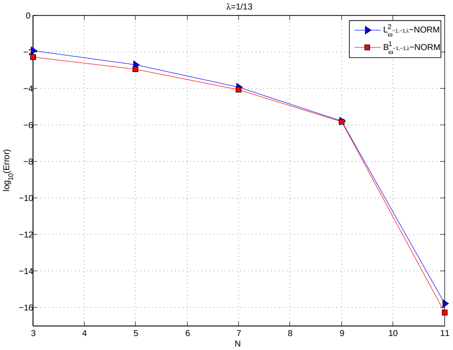


Figure:10 $u(x) = x^{12/13} - x$ with $\lambda = 1/13$.

Concluding remarks

- We have developed and analyzed a fractional spectral method for a kind of fractional integro-differential equations.

- The proposed method makes use of the fractional polynomials, also known as Müntz polynomials, constructed through a transformation of the traditional Jacobi polynomials.

- If λ is taken to be $1/q$ with q being integer, then the Müntz polynomial space $\{P_N^\lambda(I) = span\{1, x^\lambda, x^{2\lambda}, \cdots, x^{N\lambda}\}$ possesses good approximation property: the best approximation to smooth functions is of exponential convergence w.r.t. N like the traditional polynomials, although the convergence is slightly slower. In fact $P_{\lfloor N/q \rfloor}(I) \subset P_N^\lambda(I)$.

- The most remarkable feature of the method is its capability to achieve spectral convergence for the solution with limited regularity.

- The choice of λ is also of importance for the efficiency of the method, which can be made according to the following strategy:

Case I: if the solution is smooth, the optimal value is $\lambda = 1$;

Case II: if μ is a rational number p/q, the best choice is $\lambda = 1/q$; if μ is an irrational number, there is no suitable value of λ to make $u(t^{1/\lambda})$ smooth. In this case, we can take $\lambda = 1/q$ with a reasonably large q such that $u(t^{1/\lambda})$ is smooth enough.

Implementation issues

- Nonlocal terms must be evaluated by using numerical quadratures. For example, in the Muntz spectral method for the integro-differential equation, evaluation of the integral term $({}_0I_t^{\mu} u_N, v_N)$ makes use of the zeros of the orthogonal polynomial and the Gauss weights associated to the nonclassical weight function $(1 - x^{\frac{1}{\lambda}})^{\mu}$.

- For the classical orthogonal polynomials, e.g. Jacobi, Laguerre, and Hermite polynomials, formulae for the coefficients in the three-term recurrence are known in closed form. However for the nonclassical weight functions, their recurrence coefficients are not explicitly known. In this case, numerical techniques such as *Stieltjes* procedure or *Chebyshev* algorithm will be used.

- *Chebyshev* algorithm consists of calculating the desired coefficients from a three-step algorithm and the moments of the underlying weight function, i.e.,

$$M_r = \int_0^1 x^r (1 - x^{\frac{1}{\lambda}})^{\mu} dx.$$

Making the variable change $x = t^{\lambda}$ gives

$$M_r = \lambda \int_0^1 t^{\lambda r + \lambda - 1} (1 - t)^{\mu} dt = \lambda B(\lambda(r+1),\ \mu + 1).$$

- As pointed in [Esmaeili et al. 2011] the calculation of the moments M_r can be numerically problematic when the number of points is large: in order to obtain the double precision entries of the matrices, one would have to perform with about 40 digits operations.

Future extensions

Possible extensions

- Higher dimensional problems

- Other equations having corner singularities

- Using Müntz polynomials in the FE framework, i.e., Müntz spectral element methods

- Make use of more general fractional polynomial space:

$$\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \cdots, x^{\lambda_N}\}.$$

Thank you !