On nonlocal Monge–Ampère equations

Pablo Raúl Stinga

Iowa State University

Fractional PDEs: Theory, Algorithms and Applications
ICERM
June 21st, 2018

The Monge–Ampère equation

Monge optimal transport problem

We want to *transport* μ onto ν (probability measures) in an *optimal way* given that the *cost* of moving x onto $y = S(x)$ is

$$|x-y|^2 = |x-S(x)|^2$$

Monge optimal transport problem

We want to *transport* μ onto ν (probability measures) in an *optimal way* given that the *cost* of moving x onto $y = S(x)$ is

$$|x - y|^2 = |x - S(x)|^2$$

Mathematical formulation. Minimize the functional

$$\mathcal{F}(S) = \int_{\mathbb{R}^n} |x - S(x)|^2 \, d\mu(x)$$

among all maps S that *transport* μ *onto* ν: for any Borel function $\psi : \mathbb{R}^n \to \mathbb{R}$

$$\int_{\mathbb{R}^n} \psi(y) \, d\nu(y) = \int_{\mathbb{R}^n} \psi(S(x)) \, d\mu(x)$$

Monge optimal transport problem

We want to *transport* μ onto ν (probability measures) in an *optimal way* given that the *cost* of moving x onto $y = S(x)$ is

$$|x - y|^2 = |x - S(x)|^2$$

Mathematical formulation. Minimize the functional

$$\mathcal{F}(S) = \int_{\mathbb{R}^n} |x - S(x)|^2 \, d\mu(x)$$

among all maps S that *transport* μ *onto* ν: for any Borel function $\psi : \mathbb{R}^n \to \mathbb{R}$

$$\int_{\mathbb{R}^n} \psi(y) \, d\nu(y) = \int_{\mathbb{R}^n} \psi(S(x)) \, d\mu(x)$$

Theorem (Brenier, 1991)

If $\mu(x) = f(x)\,dx$ and $\nu(y)$ have finite second moments then there exists a μ-a.e. unique optimal transport map T.

Monge optimal transport problem

We want to *transport* μ onto ν (probability measures) in an *optimal way* given that the *cost* of moving x onto $y = S(x)$ is

$$|x - y|^2 = |x - S(x)|^2$$

Mathematical formulation. Minimize the functional

$$\mathcal{F}(S) = \int_{\mathbb{R}^n} |x - S(x)|^2 \, d\mu(x)$$

among all maps S that *transport* μ *onto* ν: for any Borel function $\psi : \mathbb{R}^n \to \mathbb{R}$

$$\int_{\mathbb{R}^n} \psi(y) \, d\nu(y) = \int_{\mathbb{R}^n} \psi(S(x)) \, d\mu(x)$$

Theorem (Brenier, 1991)

If $\mu(x) = f(x)\,dx$ and $\nu(y)$ have finite second moments then there exists a μ-a.e. unique optimal transport map T.

Moreover, there exists a l.s.c. **convex** *function φ, differentiable μ-a.e. such that*

$$T = \nabla\varphi \qquad \mu\text{-a.e.}$$

Optimal transport and Monge–Ampère equation

Suppose $\mu = f(x)\,dx$ and $\nu = g(y)\,dy$

Optimal transport and Monge–Ampère equation

Suppose $\mu = f(x)\,dx$ and $\nu = g(y)\,dy$

If the optimal transport map T is a diffeomorphism then, by changing variables,

$$\int_{\mathbb{R}^n} \psi(T(x))f(x)\,dx = \int_{\mathbb{R}^n} \psi(y)g(y)\,dy = \int_{\mathbb{R}^n} \psi(T(x))g(T(x))|\det \nabla T(x)|\,dx$$

Optimal transport and Monge–Ampère equation

Suppose $\mu = f(x)\,dx$ and $\nu = g(y)\,dy$

If the optimal transport map T is a diffeomorphism then, by changing variables,

$$\int_{\mathbb{R}^n} \psi(T(x))f(x)\,dx = \int_{\mathbb{R}^n} \psi(y)g(y)\,dy = \int_{\mathbb{R}^n} \psi(T(x))g(T(x))|\det \nabla T(x)|\,dx$$

Since ψ was arbitrary,

$$g(T(x))|\det \nabla T(x)| = f(x)$$

Optimal transport and Monge–Ampère equation

Suppose $\mu = f(x)\,dx$ and $\nu = g(y)\,dy$

If the optimal transport map T is a diffeomorphism then, by changing variables,

$$\int_{\mathbb{R}^n} \psi(T(x))f(x)\,dx = \int_{\mathbb{R}^n} \psi(y)g(y)\,dy = \int_{\mathbb{R}^n} \psi(T(x))g(T(x))|\det \nabla T(x)|\,dx$$

Since ψ was arbitrary,

$$g(T(x))|\det \nabla T(x)| = f(x)$$

Recall from Brenier that $T = \nabla\varphi$ for φ **convex**, so that $\nabla T = D^2\varphi > 0$ and

$$\det(D^2\varphi) = \frac{f}{g \circ \nabla\varphi}$$

Optimal transport and Monge–Ampère equation

Suppose $\mu = f(x)\,dx$ and $\nu = g(y)\,dy$

If the optimal transport map T is a diffeomorphism then, by changing variables,

$$\int_{\mathbb{R}^n} \psi(T(x))f(x)\,dx = \int_{\mathbb{R}^n} \psi(y)g(y)\,dy = \int_{\mathbb{R}^n} \psi(T(x))g(T(x))|\det \nabla T(x)|\,dx$$

Since ψ was arbitrary,

$$g(T(x))|\det \nabla T(x)| = f(x)$$

Recall from Brenier that $T = \nabla\varphi$ for φ **convex**, so that $\nabla T = D^2\varphi > 0$ and

$$\det(D^2\varphi) = \frac{f}{g \circ \nabla\varphi}$$

- The **fully nonlinear** equation

$$\det(D^2\varphi) = F$$

  is the **Monge–Ampère (MA) equation**

Convex solutions and ellipticity

Let φ be a solution to

$$\det(D^2\varphi) = F$$

Convex solutions and ellipticity

Let φ be a solution to

$$\det(D^2\varphi) = F$$

Equation for a directional derivative $\partial_e\varphi$ of the solution

$$\operatorname{trace}\big(\det(D^2\varphi)(D^2\varphi)^{-1}D^2(\partial_e\varphi)\big) = \partial_e F$$

Convex solutions and ellipticity

Let φ be a solution to

$$\det(D^2\varphi) = F$$

Equation for a directional derivative $\partial_e\varphi$ of the solution

$$\operatorname{trace}\big(\det(D^2\varphi)(D^2\varphi)^{-1}D^2(\partial_e\varphi)\big) = \partial_e F$$

Here $A_\varphi(x) = \det(D^2\varphi(x))(D^2\varphi(x))^{-1}$ is the matrix of cofactors of $D^2\varphi$

Convex solutions and ellipticity

Let φ be a solution to

$$\det(D^2\varphi) = F$$

Equation for a directional derivative $\partial_e\varphi$ of the solution

$$\operatorname{trace}\left(\det(D^2\varphi)(D^2\varphi)^{-1}D^2(\partial_e\varphi)\right) = \partial_e F$$

Here $A_\varphi(x) = \det(D^2\varphi(x))(D^2\varphi(x))^{-1}$ is the matrix of cofactors of $D^2\varphi$

If we call

$$u = \partial_e\varphi \qquad G = \partial_e F$$

then u solves the **linearized MA equation**

$$L^\varphi(u) = \operatorname{trace}(A_\varphi(x)D^2u) = G$$

Convex solutions and ellipticity

Let φ be a solution to

$$\det(D^2\varphi) = F$$

Equation for a directional derivative $\partial_e\varphi$ of the solution

$$\operatorname{trace}\left(\det(D^2\varphi)(D^2\varphi)^{-1}D^2(\partial_e\varphi)\right) = \partial_e F$$

Here $A_\varphi(x) = \det(D^2\varphi(x))(D^2\varphi(x))^{-1}$ is the matrix of cofactors of $D^2\varphi$

If we call

$$u = \partial_e\varphi \qquad G = \partial_e F$$

then u solves the **linearized MA equation**

$$L^\varphi(u) = \operatorname{trace}(A_\varphi(x)D^2u) = G$$

Linearized MA is an **elliptic equation** as soon as

$$D^2\varphi(x) > 0 \text{ (convex!)} \quad \text{and} \quad F > 0$$

Convex solutions and ellipticity

Let φ be a solution to

$$\det(D^2\varphi) = F$$

Equation for a directional derivative $\partial_e\varphi$ of the solution

$$\operatorname{trace}\left(\det(D^2\varphi)(D^2\varphi)^{-1}D^2(\partial_e\varphi)\right) = \partial_e F$$

Here $A_\varphi(x) = \det(D^2\varphi(x))(D^2\varphi(x))^{-1}$ is the matrix of cofactors of $D^2\varphi$

If we call

$$u = \partial_e\varphi \qquad G = \partial_e F$$

then u solves the **linearized MA equation**

$$L^\varphi(u) = \operatorname{trace}(A_\varphi(x)D^2u) = G$$

Linearized MA is an **elliptic equation** as soon as

$$D^2\varphi(x) > 0 \textbf{ (convex!)} \quad \text{and} \quad F > 0$$

- MA equation is **degenerate elliptic**.

The MA geometry

There is an intrinsic quasi-metric space associated with MA

The MA geometry

There is an intrinsic quasi-metric space associated with MA

- **Uniformly elliptic.** $L(u) = \mathrm{trace}(A(x)D^2u)$ with $A(x) \sim I$

The MA geometry

There is an intrinsic quasi-metric space associated with MA

- **Uniformly elliptic.** $L(u) = \mathrm{trace}(A(x)D^2u)$ with $A(x) \sim I$
  P quadratic polynomial, then $L(P) \approx 1$

The MA geometry

There is an intrinsic quasi-metric space associated with MA

- **Uniformly elliptic.** $L(u) = \text{trace}(A(x)D^2u)$ with $A(x) \sim I$
  P quadratic polynomial, then $L(P) \approx 1$
  Sublevel sets of P are all the Euclidean balls.

The MA geometry

There is an intrinsic quasi-metric space associated with MA

- **Uniformly elliptic.** $L(u) = \mathrm{trace}(A(x)D^2u)$ with $A(x) \sim I$

  P quadratic polynomial, then $L(P) \approx 1$

  Sublevel sets of P are all the Euclidean balls. Harnack inequality in balls

The MA geometry

There is an intrinsic quasi-metric space associated with MA

- **Uniformly elliptic.** $L(u) = \mathrm{trace}(A(x)D^2u)$ with $A(x) \sim I$
  P quadratic polynomial, then $L(P) \approx 1$
  Sublevel sets of P are all the Euclidean balls. Harnack inequality in balls
- **Linearized MA.** $L^{\varphi}(u) = \mathrm{trace}(A_{\varphi}(x)D^2u)$ with $\det(D^2\varphi) \approx 1$

The MA geometry

There is an intrinsic quasi-metric space associated with MA

- **Uniformly elliptic.** $L(u) = \mathrm{trace}(A(x)D^2u)$ with $A(x) \sim I$
  P quadratic polynomial, then $L(P) \approx 1$
  Sublevel sets of P are all the Euclidean balls. Harnack inequality in balls

- **Linearized MA.** $L^\varphi(u) = \mathrm{trace}(A_\varphi(x)D^2u)$ with $\det(D^2\varphi) \approx 1$
  ℓ linear function, then $L^\varphi(\varphi - \ell) = L^\varphi(\varphi) \approx 1$

The MA geometry

There is an intrinsic quasi-metric space associated with MA

- **Uniformly elliptic.** $L(u) = \mathrm{trace}(A(x)D^2u)$ with $A(x) \sim I$
  P quadratic polynomial, then $L(P) \approx 1$
  Sublevel sets of P are all the Euclidean balls. Harnack inequality in balls
- **Linearized MA.** $L^\varphi(u) = \mathrm{trace}(A_\varphi(x)D^2u)$ with $\det(D^2\varphi) \approx 1$
  ℓ linear function, then $L^\varphi(\varphi - \ell) = L^\varphi(\varphi) \approx 1$
  The functions $\varphi - \ell$ play the same role of P as above

The MA geometry

There is an intrinsic quasi-metric space associated with MA

- **Uniformly elliptic.** $L(u) = \mathrm{trace}(A(x)D^2u)$ with $A(x) \sim I$
  P quadratic polynomial, then $L(P) \approx 1$
  Sublevel sets of P are all the Euclidean balls. Harnack inequality in balls
- **Linearized MA.** $L^\varphi(u) = \mathrm{trace}(A_\varphi(x)D^2u)$ with $\det(D^2\varphi) \approx 1$
  ℓ linear function, then $L^\varphi(\varphi - \ell) = L^\varphi(\varphi) \approx 1$
  The functions $\varphi - \ell$ play the same role of P as above
  The geometry is given by the sublevel sets of $\varphi - \ell$ or **sections of** φ

The MA geometry

There is an intrinsic quasi-metric space associated with MA

- **Uniformly elliptic.** $L(u) = \mathrm{trace}(A(x)D^2u)$ with $A(x) \sim I$
  P quadratic polynomial, then $L(P) \approx 1$
  Sublevel sets of P are all the Euclidean balls. Harnack inequality in balls

- **Linearized MA.** $L^\varphi(u) = \mathrm{trace}(A_\varphi(x)D^2u)$ with $\det(D^2\varphi) \approx 1$
  ℓ linear function, then $L^\varphi(\varphi - \ell) = L^\varphi(\varphi) \approx 1$
  The functions $\varphi - \ell$ play the same role of P as above
  The geometry is given by the sublevel sets of $\varphi - \ell$ or **sections of** φ

  MA quasi-metric.

$$\delta_\varphi(x_0, x) = \varphi(x) - \varphi(x_0) - \nabla\varphi(x_0) \cdot (x - x_0)$$

  MA sections.

$$S_\varphi(x_0, R) = \left\{x \in \mathbb{R}^n : \delta_\varphi(x_0, x) < R\right\}$$

The MA geometry

There is an intrinsic quasi-metric space associated with MA

- **Uniformly elliptic.** $L(u) = \mathrm{trace}(A(x)D^2u)$ with $A(x) \sim I$
  P quadratic polynomial, then $L(P) \approx 1$
  Sublevel sets of P are all the Euclidean balls. Harnack inequality in balls

- **Linearized MA.** $L^\varphi(u) = \mathrm{trace}(A_\varphi(x)D^2u)$ with $\det(D^2\varphi) \approx 1$
  ℓ linear function, then $L^\varphi(\varphi - \ell) = L^\varphi(\varphi) \approx 1$
  The functions $\varphi - \ell$ play the same role of P as above
  The geometry is given by the sublevel sets of $\varphi - \ell$ or **sections of** φ

  MA quasi-metric.

$$\delta_\varphi(x_0, x) = \varphi(x) - \varphi(x_0) - \nabla\varphi(x_0) \cdot (x - x_0)$$

  MA sections.

$$S_\varphi(x_0, R) = \{x \in \mathbb{R}^n : \delta_\varphi(x_0, x) < R\}$$

- If $\varphi(x) = |x|^2/2$ then $L^\varphi = \Delta$ and $S_\varphi(x_0, R) = B(x_0, \sqrt{2R})$

Harnack inequality for linearized MA

Assumption. The measure $\mu = \det(D^2\varphi) > 0$ satisfies μ_∞-**condition**: given $0 < \delta_1 < 1$ there exists $0 < \delta_2 < 1$ such that for all sections S and all $E \subset S$,

$$|E| < \delta_2 |S| \quad \text{implies} \quad \mu(E) < \delta_1 \mu(S)$$

Harnack inequality for linearized MA

Assumption. The measure $\mu = \det(D^2\varphi) > 0$ satisfies μ_∞-**condition**: given $0 < \delta_1 < 1$ there exists $0 < \delta_2 < 1$ such that for all sections S and all $E \subset S$,

$$|E| < \delta_2 |S| \quad \text{implies} \quad \mu(E) < \delta_1 \mu(S)$$

Theorem (Caffarelli–Gutiérrez, *Amer. J. Math* 1997)

There exist geometric constants $C, K > 1$ *and* $0 < \tau < 1$ *such that for any section* $S_0 = S_\varphi(x_0, R_0)$, *every solution to*

$$\begin{cases} L^\varphi u = 0 & \text{in } S_0 \\ u \geq 0 & \text{in } S_0 \end{cases}$$

and every section $S_\varphi(x, KR) \subset\subset S_0$, *we have*

$$\sup_{S_\varphi(x,\tau R)} u \leq C \inf_{S_\varphi(x,\tau R)} u$$

Harnack inequality for linearized MA

Assumption. The measure $\mu = \det(D^2\varphi) > 0$ satisfies μ_∞-**condition**: given $0 < \delta_1 < 1$ there exists $0 < \delta_2 < 1$ such that for all sections S and all $E \subset S$,

$$|E| < \delta_2 |S| \quad \text{implies} \quad \mu(E) < \delta_1 \mu(S)$$

Theorem (Caffarelli–Gutiérrez, *Amer. J. Math* 1997)

There exist geometric constants $C, K > 1$ and $0 < \tau < 1$ such that for any section $S_0 = S_\varphi(x_0, R_0)$, every solution to

$$\begin{cases} L^\varphi u = 0 & \text{in } S_0 \\ u \geq 0 & \text{in } S_0 \end{cases}$$

and every section $S_\varphi(x, KR) \subset\subset S_0$, we have

$$\sup_{S_\varphi(x,\tau R)} u \leq C \inf_{S_\varphi(x,\tau R)} u$$

In particular, there exists $0 < \alpha < 1$ such that if $L^\varphi u = 0$ then

$$|u(x) - u(z)| \leq C\delta_\varphi(x,z)^\alpha \quad \text{for any } z \in S_\varphi(x, R)$$

Fractional linearized MA equation

- Maldonado–Stinga, Harnack inequality for the fractional nonlocal linearized Monge–Ampère equation, *Calc. Var. PDE* (2017)

The fractional linearized MA operator

For $\varphi \in C^3$ with $D^2\varphi > 0$ and a section S of φ we consider

$$\begin{cases} L^{\varphi} u = -\operatorname{trace}(A_{\varphi}(x) D^2 u) & \text{in } S \\ u = 0 & \text{on } \partial S \end{cases}$$

Dirichlet linearized MA operator. The operator is nonvariational.

The fractional linearized MA operator

For $\varphi \in C^3$ with $D^2\varphi > 0$ and a section S of φ we consider

$$\begin{cases} L^\varphi u = -\operatorname{trace}(A_\varphi(x)D^2 u) & \text{in } S \\ u = 0 & \text{on } \partial S \end{cases}$$

Dirichlet linearized MA operator. The operator is nonvariational.

We want to define $(L^\varphi)^s$ for $0 < s < 1$.

The fractional linearized MA operator

For $\varphi \in C^3$ with $D^2\varphi > 0$ and a section S of φ we consider

$$\begin{cases} L^{\varphi} u = -\operatorname{trace}(A_{\varphi}(x) D^2 u) & \text{in } S \\ u = 0 & \text{on } \partial S \end{cases}$$

Dirichlet linearized MA operator. The operator is nonvariational.

We want to define $(L^{\varphi})^s$ for $0 < s < 1$.

We define fractional powers L^s with the method of semigroups

The fractional linearized MA operator

For $\varphi \in C^3$ with $D^2\varphi > 0$ and a section S of φ we consider

$$\begin{cases} L^\varphi u = -\operatorname{trace}(A_\varphi(x)D^2u) & \text{in } S \\ u = 0 & \text{on } \partial S \end{cases}$$

Dirichlet linearized MA operator. The operator is nonvariational.

We want to define $(L^\varphi)^s$ for $0 < s < 1$.

We define fractional powers L^s with the method of semigroups

Let $v(x,t) = e^{-tL}u(x)$ be the **heat semigroup** generated by L:

$$\begin{cases} \partial_t v = -Lv & \text{for } t > 0 \\ v(x,0) = u(x) \end{cases}$$

The fractional linearized MA operator

For $\varphi \in C^3$ with $D^2\varphi > 0$ and a section S of φ we consider

$$\begin{cases} L^\varphi u = -\operatorname{trace}(A_\varphi(x) D^2 u) & \text{in } S \\ u = 0 & \text{on } \partial S \end{cases}$$

Dirichlet linearized MA operator. The operator is nonvariational.
We want to define $(L^\varphi)^s$ for $0 < s < 1$.

We define fractional powers L^s with the method of semigroups

Let $v(x,t) = e^{-tL}u(x)$ be the **heat semigroup** generated by L:

$$\begin{cases} \partial_t v = -Lv & \text{for } t > 0 \\ v(x,0) = u(x) \end{cases}$$

For $0 < s < 1$,

$$L^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-tL}u(x) - u(x) \right) \frac{dt}{t^{1+s}}$$

Fractional operators

$$L^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-tL}u(x) - u(x)\right) \frac{dt}{t^{1+s}}$$

This identity comes from a numerical formula with the Gamma function

- For $\lambda \geq 0$,

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+s}}$$

Fractional operators

$$L^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-tL}u(x) - u(x)\right) \frac{dt}{t^{1+s}}$$

This identity comes from a numerical formula with the Gamma function

- For $\lambda \geq 0$,

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+s}}$$

For example,

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta}u(x) - u(x)\right) \frac{dt}{t^{1+s}}$$

$$= c_{n,s} \, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x - z|^{n+2s}} \, dz$$

- Stinga–Torrea, Extension problem and Harnack's inequality for some fractional operators, *Comm. PDE* (2010) **(Hilbert spaces – variational)**

The fractional linearized MA operator

For **nonvariational operators** we use the **semigroup method** from

- Galé–Miana–Stinga, Extension problem and fractional operators: semigroups and wave equations, *J. Evol. Equ.* (2013) **(Banach spaces – nonvariational)**

The fractional linearized MA operator

For **nonvariational operators** we use the **semigroup method** from

- Galé–Miana–Stinga, Extension problem and fractional operators: semigroups and wave equations, *J. Evol. Equ.* (2013) **(Banach spaces – nonvariational)**

For $0 < s < 1$ we define the **fractional linearlized MA operator** as

$$(L^{\varphi})^s u(x) = \frac{1}{\Gamma(-s)} \int_0^{\infty} \left(e^{-tL^{\varphi}} u(x) - u(x) \right) \frac{dt}{t^{1+s}}$$

where $v(x,t) = e^{-tL^{\varphi}} u(x)$ is the solution to

$$\begin{cases} \partial_t v = -L^{\varphi} v & \text{in } S \times (0,\infty) \\ v = 0 & \text{on } \partial S \times [0,\infty) \\ v(x,0) = u(x) & \text{on } S \end{cases}$$

The fractional linearized MA operator

For **nonvariational operators** we use the **semigroup method** from

- Galé–Miana–Stinga, Extension problem and fractional operators: semigroups and wave equations, *J. Evol. Equ.* (2013) **(Banach spaces – nonvariational)**

For $0 < s < 1$ we define the **fractional linearlized MA operator** as

$$(L^{\varphi})^s u(x) = \frac{1}{\Gamma(-s)} \int_0^{\infty} \left(e^{-tL^{\varphi}} u(x) - u(x)\right) \frac{dt}{t^{1+s}}$$

where $v(x,t) = e^{-tL^{\varphi}} u(x)$ is the solution to

$$\begin{cases} \partial_t v = -L^{\varphi} v & \text{in } S \times (0,\infty) \\ v = 0 & \text{on } \partial S \times [0,\infty) \\ v(x,0) = u(x) & \text{on } S \end{cases}$$

- The semigroup $e^{-tL^{\varphi}}$ has a heat kernel.

The fractional linearized MA operator

For **nonvariational operators** we use the **semigroup method** from

- Galé–Miana–Stinga, Extension problem and fractional operators: semigroups and wave equations, *J. Evol. Equ.* (2013) **(Banach spaces – nonvariational)**

For $0 < s < 1$ we define the **fractional linearlized MA operator** as

$$(L^{\varphi})^s u(x) = \frac{1}{\Gamma(-s)} \int_0^{\infty} \left(e^{-tL^{\varphi}} u(x) - u(x)\right) \frac{dt}{t^{1+s}}$$

where $v(x,t) = e^{-tL^{\varphi}} u(x)$ is the solution to

$$\begin{cases} \partial_t v = -L^{\varphi} v & \text{in } S \times (0,\infty) \\ v = 0 & \text{on } \partial S \times [0,\infty) \\ v(x,0) = u(x) & \text{on } S \end{cases}$$

- The semigroup $e^{-tL^{\varphi}}$ has a heat kernel.
- One can see that

$$(L^{\varphi})^s u(x) = \mathrm{P.V.} \int_S (u(x) - u(z)) K_s^{\varphi}(x,z)\, dz + B_s^{\varphi}(x) u(x)$$

Harnack inequality

Assumption. The measure $\mu = \det(D^2\varphi) > 0$ satisfies the **doubling condition**: there exists $C_d \geq 1$ such that for any section S of φ we have

$$\mu(S) \leq C_d \mu(\tfrac{1}{2}S)$$

Harnack inequality

Assumption. The measure $\mu = \det(D^2\varphi) > 0$ satisfies the **doubling condition**: there exists $C_d \geq 1$ such that for any section S of φ we have

$$\mu(S) \leq C_d \mu(\tfrac{1}{2}S)$$

Theorem (Maldonado–Stinga, 2017)

There exist geometric constants $C, K > 1$ and $0 < \tau < 1$ such that for any section S_0 of φ, every $f \in C_0(\overline{S_0})$, every solution u to

$$\begin{cases} (L^\varphi)^s u = f & \text{in } S_0 \\ u \geq 0 & \text{in } S_0 \end{cases}$$

and every section $S_\varphi(x, KR) \subset\subset S_0$,

$$\sup_{S_\varphi(x,\tau R)} u \leq C\Big(\inf_{S_\varphi(x,\tau R)} u + R^s \|f\|_{L^\infty(S_\varphi(x,KR))}\Big)$$

Harnack inequality

Assumption. The measure $\mu = \det(D^2\varphi) > 0$ satisfies the **doubling condition**: there exists $C_d \geq 1$ such that for any section S of φ we have

$$\mu(S) \leq C_d \mu(\tfrac{1}{2}S)$$

Theorem (Maldonado–Stinga, 2017)

There exist geometric constants $C, K > 1$ and $0 < \tau < 1$ such that for any section S_0 of φ, every $f \in C_0(\overline{S_0})$, every solution u to

$$\begin{cases} (L^\varphi)^s u = f & \text{in } S_0 \\ u \geq 0 & \text{in } S_0 \end{cases}$$

and every section $S_\varphi(x, KR) \subset\subset S_0$,

$$\sup_{S_\varphi(x,\tau R)} u \leq C\Big(\inf_{S_\varphi(x,\tau R)} u + R^s \|f\|_{L^\infty(S_\varphi(x,KR))}\Big)$$

As a consequence, there exists $0 < \alpha < 1$ such that if $(L^\varphi)^s u = f$ then

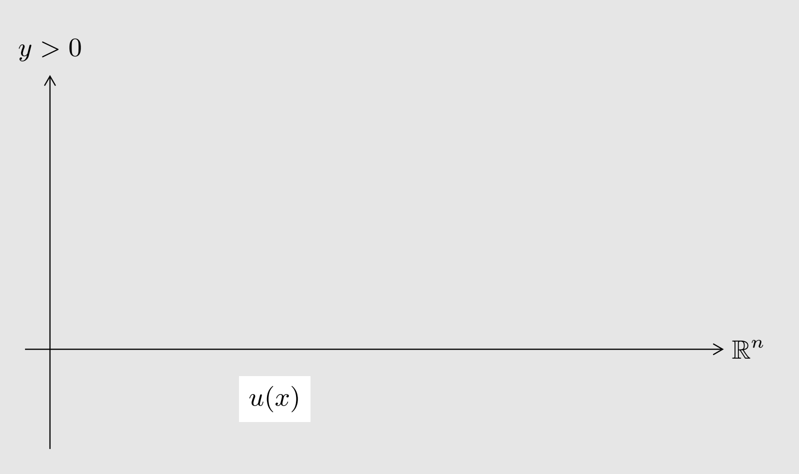
$$|u(x) - u(z)| \leq C\delta_\varphi(x,z)^\alpha \quad \text{for any } z \in S_\varphi(x, R)$$

Caffarelli–Silvestre extension problem (2007)

Aim. Describe $(-\Delta)^s$ (nonlocal in $\mathbb{R}^n$) with local PDEs

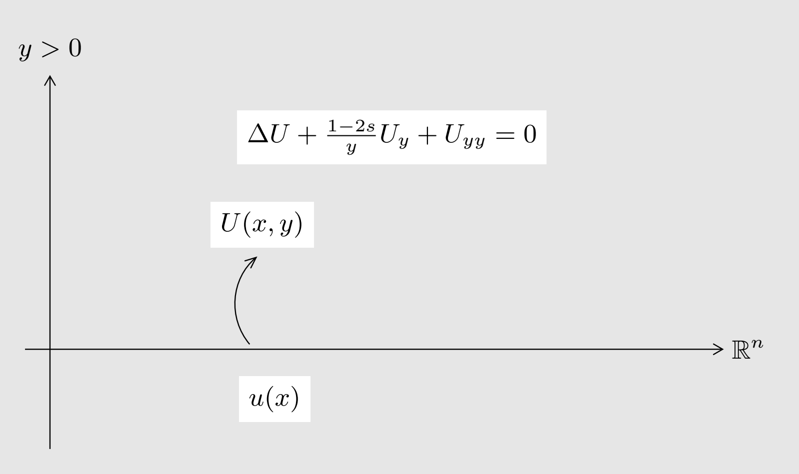
Caffarelli–Silvestre extension problem (2007)

Aim. Describe $(-\Delta)^s$ (nonlocal in $\mathbb{R}^n$) with local PDEs



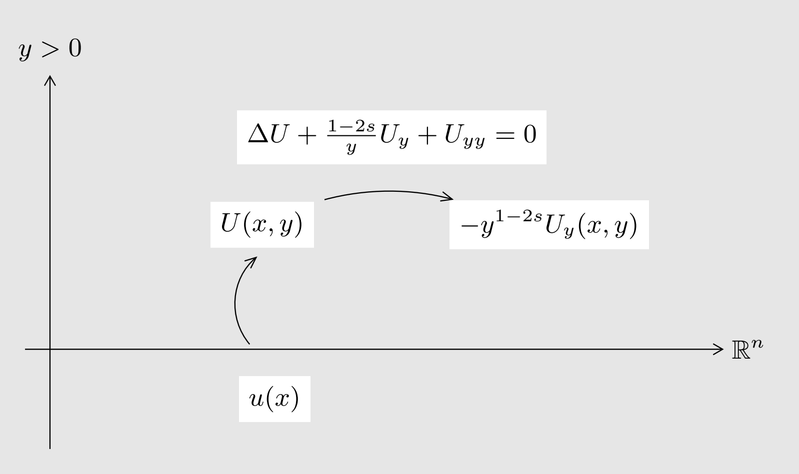
Caffarelli–Silvestre extension problem (2007)

Aim. Describe $(-\Delta)^s$ (nonlocal in $\mathbb{R}^n$) with local PDEs



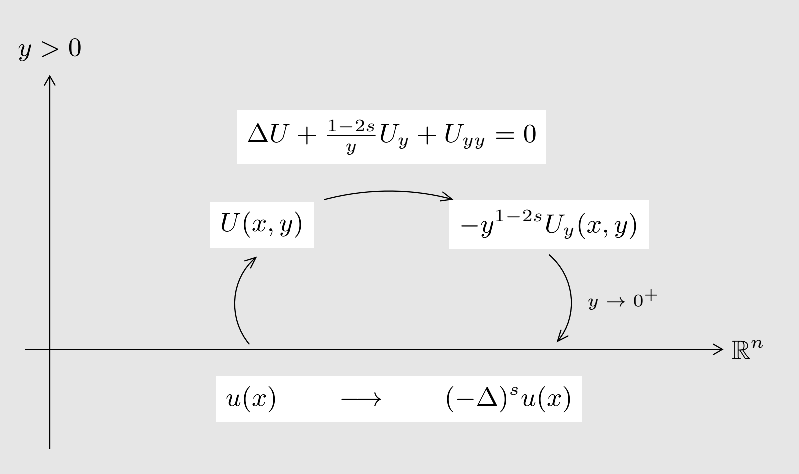
Caffarelli–Silvestre extension problem (2007)

Aim. Describe $(-\Delta)^s$ (nonlocal in $\mathbb{R}^n$) with local PDEs



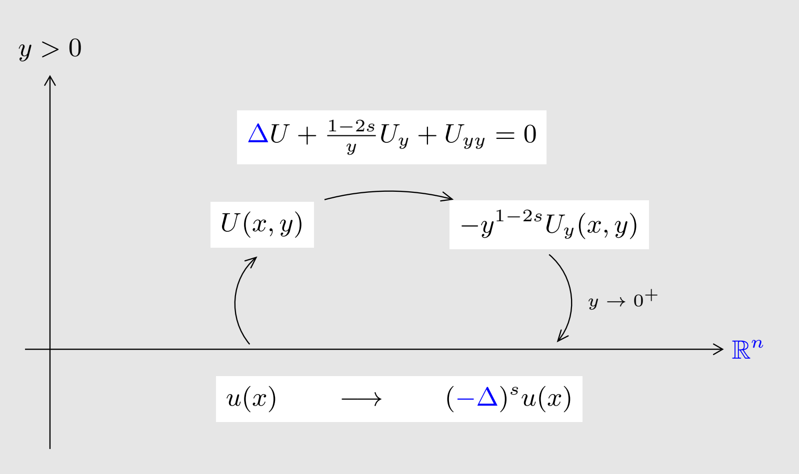
Caffarelli–Silvestre extension problem (2007)

Aim. Describe $(-\Delta)^s$ (nonlocal in $\mathbb{R}^n$) with local PDEs



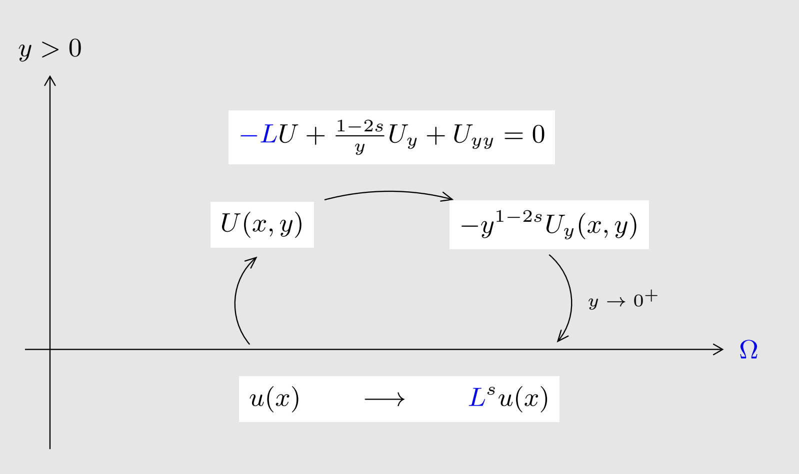
Caffarelli–Silvestre extension problem (2007)

Aim. Describe $(-\Delta)^s$ (nonlocal in $\mathbb{R}^n$) with local PDEs



Stinga–Torrea (2010) and Galé–Miana–Stinga (2013)

Aim. Describe L^s (nonlocal in Ω) with local PDEs



Extension for fractional linearized MA

The extension problem for $(L^\varphi)^s$ **in nondivergence form** reads

$$\begin{cases} \operatorname{trace}(A_\varphi(x)D^2U) + z^{2-1/s}U_{zz} = 0 & \text{for } x \in S, z > 0 \\ U = 0 & \text{for } x \in \partial S, z \geq 0 \\ -U_z\big|_{z=0^+} = f(x) & \text{for } x \in S \end{cases}$$

Then

$$(L^\varphi)^s u = f \quad \text{if and only if} \quad U(x,0) = u(x)$$

Extension for fractional linearized MA

The extension problem for $(L^{\varphi})^s$ **in nondivergence form** reads

$$\begin{cases} \operatorname{trace}(A_{\varphi}(x)D^2U) + z^{2-1/s}U_{zz} = 0 & \text{for } x \in S, z > 0 \\ U = 0 & \text{for } x \in \partial S, z \geq 0 \\ -U_z\big|_{z=0^+} = f(x) & \text{for } x \in S \end{cases}$$

Then

$$(L^{\varphi})^s u = f \quad \text{if and only if} \quad U(x,0) = u(x)$$

The extension equation is a **linearized MA equation**: for $\tilde{U}(x,z) = U(x,|z|)$,

$$\operatorname{trace}(A_{\varphi}(x)D^2\tilde{U}) + |z|^{2-1/s}\tilde{U}_{zz} = \operatorname{trace}(A_{\Phi}(x,z)D^2_{x,z}\tilde{U}) = 0$$

for $z \neq 0$, where $\Phi(x,z) = \varphi(x) + \frac{s^2}{(1-s)}|z|^{1/s}$

In addition, $\mu_{\Phi} = \det(D^2\Phi)$ satisfies the **doubling condition**

Extension for fractional linearized MA

The extension problem for $(L^{\varphi})^s$ **in nondivergence form** reads

$$\begin{cases} \operatorname{trace}(A_{\varphi}(x)D^2U) + z^{2-1/s}U_{zz} = 0 & \text{for } x \in S, z > 0 \\ U = 0 & \text{for } x \in \partial S, z \geq 0 \\ -U_z\big|_{z=0^+} = f(x) & \text{for } x \in S \end{cases}$$

Then

$$(L^{\varphi})^s u = f \quad \text{if and only if} \quad U(x,0) = u(x)$$

The extension equation is a **linearized MA equation**: for $\tilde{U}(x,z) = U(x,|z|)$,

$$\operatorname{trace}(A_{\varphi}(x)D^2\tilde{U}) + |z|^{2-1/s}\tilde{U}_{zz} = \operatorname{trace}(A_{\Phi}(x,z)D^2_{x,z}\tilde{U}) = 0$$

for $z \neq 0$, where $\Phi(x,z) = \varphi(x) + \frac{s^2}{(1-s)}|z|^{1/s}$

In addition, $\mu_{\Phi} = \det(D^2\Phi)$ satisfies the **doubling condition**

BUT still there is a degeneracy/singularity of $D^2\Phi$ at $z = 0$**!**

Nondivergence meets divergence

The columns of $A_\varphi(x) = \det(D^2\varphi(x))(D^2\varphi(x))^{-1}$ are divergence free.

Nondivergence meets divergence

The columns of $A_\varphi(x) = \det(D^2\varphi(x))(D^2\varphi(x))^{-1}$ are divergence free.

Then linearized MA also has **divergence structure**:

$$-\operatorname{trace}(A_\varphi(x)D^2u) = -\operatorname{div}(A_\varphi(x)\nabla u) \equiv \mathcal{L}^\varphi u$$

Nondivergence meets divergence

The columns of $A_\varphi(x) = \det(D^2\varphi(x))(D^2\varphi(x))^{-1}$ are divergence free.

Then linearized MA also has **divergence structure**:

$$-\operatorname{trace}(A_\varphi(x)D^2u) = -\operatorname{div}(A_\varphi(x)\nabla u) \equiv \mathcal{L}^\varphi u$$

With a change of variables $z \longleftrightarrow y$ the extension equation becomes variational

$$\operatorname{trace}(A_\varphi(x)D^2U) + z^{2-1/s}U_{zz} = 0 \quad \longleftrightarrow \quad \operatorname{div}(y^{1-2s}A_\varphi(x)\nabla_{x,y}V) = 0$$

Nondivergence meets divergence

The columns of $A_\varphi(x) = \det(D^2\varphi(x))(D^2\varphi(x))^{-1}$ are divergence free.

Then linearized MA also has **divergence structure**:

$$-\operatorname{trace}(A_\varphi(x)D^2u) = -\operatorname{div}(A_\varphi(x)\nabla u) \equiv \mathcal{L}^\varphi u$$

With a change of variables $z \longleftrightarrow y$ the extension equation becomes variational

$$\operatorname{trace}(A_\varphi(x)D^2U) + z^{2-1/s}U_{zz} = 0 \quad \longleftrightarrow \quad \operatorname{div}(y^{1-2s}A_\varphi(x)\nabla_{x,y}V) = 0$$

The variational side will give us weak Harnack inequality $(|\{z=0\}| = 0)$

Nondivergence meets divergence

The columns of $A_\varphi(x) = \det(D^2\varphi(x))(D^2\varphi(x))^{-1}$ are divergence free.

Then linearized MA also has **divergence structure**:

$$-\operatorname{trace}(A_\varphi(x)D^2u) = -\operatorname{div}(A_\varphi(x)\nabla u) \equiv \mathcal{L}^\varphi u$$

With a change of variables $z \longleftrightarrow y$ the extension equation becomes variational

$$\operatorname{trace}(A_\varphi(x)D^2U) + z^{2-1/s}U_{zz} = 0 \quad \longleftrightarrow \quad \operatorname{div}(y^{1-2s}A_\varphi(x)\nabla_{x,y}V) = 0$$

The variational side will give us weak Harnack inequality $(|\{z=0\}| = 0)$

$\mathcal{L}^\varphi$ has a sequence of eigenvalues/eigenfunctions (λ_k, ψ_k), so we can define

$$(\mathcal{L}^\varphi)^s u = \sum \lambda_k^s u_k \psi_k$$

Nondivergence meets divergence

The columns of $A_\varphi(x) = \det(D^2\varphi(x))(D^2\varphi(x))^{-1}$ are divergence free.

Then linearized MA also has **divergence structure**:

$$-\operatorname{trace}(A_\varphi(x)D^2u) = -\operatorname{div}(A_\varphi(x)\nabla u) \equiv \mathcal{L}^\varphi u$$

With a change of variables $z \longleftrightarrow y$ the extension equation becomes variational

$$\operatorname{trace}(A_\varphi(x)D^2U) + z^{2-1/s}U_{zz} = 0 \quad \longleftrightarrow \quad \operatorname{div}(y^{1-2s}A_\varphi(x)\nabla_{x,y}V) = 0$$

The variational side will give us weak Harnack inequality $(|\{z=0\}| = 0)$

$\mathcal{L}^\varphi$ has a sequence of eigenvalues/eigenfunctions (λ_k, ψ_k), so we can define

$$(\mathcal{L}^\varphi)^s u = \sum \lambda_k^s u_k \psi_k$$

Theorem (Maldonado–Stinga, 2017)

$$(L^\varphi)^s = (\mathcal{L}^\varphi)^s$$

Obstacle problem for a fractional MA equation

- Jhaveri–Stinga, The obstacle problem for a fractional Monge–Ampère equation, *arXiv* (2017)

MA is an extremal operator

For u convex and C^2 we have

$$n\det(D^2u(x))^{1/n} = \inf\left\{\Delta(u\circ A)(A^{-1}x) : A = A^T > 0, \det(A) = 1\right\}$$
$$= \inf\left\{\operatorname{trace}(A^2D^2u(x)) : A = A^T > 0, \det(A) = 1\right\}$$

MA is an extremal operator

For u convex and C^2 we have

$$n\det(D^2u(x))^{1/n} = \inf\left\{\Delta(u\circ A)(A^{-1}x) : A = A^T > 0, \det(A) = 1\right\}$$
$$= \inf\left\{\operatorname{trace}(A^2D^2u(x)) : A = A^T > 0, \det(A) = 1\right\}$$

Infimum is achieved by $A^2 = \det(D^2u)^{1/n}(D^2u)^{-1}$

MA is an extremal operator

For u convex and C^2 we have

$$n\det(D^2u(x))^{1/n} = \inf\left\{\Delta(u\circ A)(A^{-1}x) : A = A^T > 0, \det(A) = 1\right\}$$
$$= \inf\left\{\operatorname{trace}(A^2D^2u(x)) : A = A^T > 0, \det(A) = 1\right\}$$

Infimum is achieved by $A^2 = \det(D^2u)^{1/n}(D^2u)^{-1}$

MA is degenerate elliptic. Matrices of the form

$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1/\varepsilon \end{pmatrix}$$

enter in the computation of the infimum.

MA is an extremal operator

For u convex and C^2 we have

$$
\begin{aligned}
n\det(D^2u(x))^{1/n} &= \inf\left\{\Delta(u\circ A)(A^{-1}x) : A = A^T > 0, \det(A) = 1\right\} \\
&= \inf\left\{\operatorname{trace}(A^2D^2u(x)) : A = A^T > 0, \det(A) = 1\right\}
\end{aligned}
$$

Infimum is achieved by $A^2 = \det(D^2u)^{1/n}(D^2u)^{-1}$

MA is degenerate elliptic. Matrices of the form

$$
A = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1/\varepsilon \end{pmatrix}
$$

enter in the computation of the infimum.

Nevertheless, if u **is convex,** $D^2_{ee}u \leq M_0$ **(semiconcave) and**

$$
\det(D^2u) = \prod_{j=1}^{n} \lambda_j = f(x) \geq \eta_0 > 0
$$

then $D^2u \sim I$. Thus $A > \lambda I$ in the computation of the infimum.

Fractional MA equation

Definition (Caffarelli–Charro, *Ann. of PDE* 2015)

For $1/2 < s < 1$ *and* u *linear at infinity,*

$$\mathcal{D}_s u(x) = \inf\left\{-(-\Delta)^s(u \circ A)(A^{-1}x) : A = A^T > 0, \det(A) = 1\right\}$$

Fractional MA equation

Definition (Caffarelli–Charro, *Ann. of PDE* 2015)

For $1/2 < s < 1$ and u linear at infinity,

$$\mathcal{D}_s u(x) = \inf\left\{-(-\Delta)^s (u \circ A)(A^{-1}x) : A = A^T > 0, \det(A) = 1\right\}$$

Integro-differential formula:

$$\mathcal{D}_s u(x) = \inf_{A>0,\det(A)=1} \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x+z) + u(x-z) - 2u(x)}{|A^{-1}z|^{n+2s}}\, dz$$

Fractional MA equation

Definition (Caffarelli–Charro, *Ann. of PDE* 2015)

For $1/2 < s < 1$ and u linear at infinity,

$$\mathcal{D}_s u(x) = \inf\left\{-(-\Delta)^s(u\circ A)(A^{-1}x) : A = A^T > 0, \det(A) = 1\right\}$$

Integro-differential formula:

$$\mathcal{D}_s u(x) = \inf_{A>0,\det(A)=1} \frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{u(x+z)+u(x-z)-2u(x)}{|A^{-1}z|^{n+2s}}\,dz$$

The fractional MA operator is **degenerate elliptic**.

Fractional MA equation

Definition (Caffarelli–Charro, *Ann. of PDE* 2015)

For $1/2 < s < 1$ *and* u *linear at infinity,*

$$\mathcal{D}_s u(x) = \inf\left\{-(-\Delta)^s(u \circ A)(A^{-1}x) : A = A^T > 0, \det(A) = 1\right\}$$

Integro-differential formula:

$$\mathcal{D}_s u(x) = \inf_{A>0,\det(A)=1} \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x+z)+u(x-z)-2u(x)}{|A^{-1}z|^{n+2s}}\,dz$$

The fractional MA operator is **degenerate elliptic**.

Theorem (Caffarelli–Charro, *Ann. of PDE* 2015)

$$\lim_{s\to 1^-} \mathcal{D}_s u(x) = n\det(D^2u(x))^{1/n}$$

Fractional MA equation

Caffarelli and Charro considered the Dirichlet problem

$$\begin{cases} \mathcal{D}_s \bar{u} = \bar{u} - \phi & \text{in } \mathbb{R}^n \\ \lim_{|x|\to\infty} (\bar{u} - \phi)(x) = 0 \end{cases}$$

where ϕ is convex and behaves like a cone at infinity.

Fractional MA equation

Caffarelli and Charro considered the Dirichlet problem

$$\begin{cases} \mathcal{D}_s \bar{u} = \bar{u} - \phi & \text{in } \mathbb{R}^n \\ \lim_{|x|\to\infty} (\bar{u} - \phi)(x) = 0 \end{cases}$$

where ϕ is convex and behaves like a cone at infinity.

In particular, $\mathcal{D}_s \phi \geq 0 = \phi - \phi$, so ϕ is a subsolution and $\bar{u} \geq \phi$.

Fractional MA equation

Caffarelli and Charro considered the Dirichlet problem

$$\begin{cases} \mathcal{D}_s \bar{u} = \bar{u} - \phi & \text{in } \mathbb{R}^n \\ \lim_{|x|\to\infty} (\bar{u} - \phi)(x) = 0 \end{cases}$$

where ϕ is convex and behaves like a cone at infinity.

In particular, $\mathcal{D}_s \phi \geq 0 = \phi - \phi$, so ϕ is a subsolution and $\bar{u} \geq \phi$.

There exists a unique viscosity solution $\bar{u}$ that is **Lipschitz** and **semiconcave**.

Fractional MA equation

Caffarelli and Charro considered the Dirichlet problem

$$\begin{cases} \mathcal{D}_s \bar{u} = \bar{u} - \phi & \text{in } \mathbb{R}^n \\ \lim_{|x|\to\infty} (\bar{u} - \phi)(x) = 0 \end{cases}$$

where ϕ is convex and behaves like a cone at infinity.

In particular, $\mathcal{D}_s \phi \geq 0 = \phi - \phi$, so ϕ is a subsolution and $\bar{u} \geq \phi$.

There exists a unique viscosity solution $\bar{u}$ that is **Lipschitz** and **semiconcave**.

Moreover, $\bar{u}$ has the crucial property

$$\bar{u} > \phi \quad \text{in } \mathbb{R}^n$$

Fractional MA equation

Caffarelli and Charro considered the Dirichlet problem

$$\begin{cases} \mathcal{D}_s \bar{u} = \bar{u} - \phi & \text{in } \mathbb{R}^n \\ \lim_{|x|\to\infty} (\bar{u} - \phi)(x) = 0 \end{cases}$$

where ϕ is convex and behaves like a cone at infinity.

In particular, $\mathcal{D}_s \phi \geq 0 = \phi - \phi$, so ϕ is a subsolution and $\bar{u} \geq \phi$.

There exists a unique viscosity solution $\bar{u}$ that is **Lipschitz** and **semiconcave**.

Moreover, $\bar{u}$ has the crucial property

$$\bar{u} > \phi \quad \text{in } \mathbb{R}^n$$

Theorem (Caffarelli–Charro, *Ann. of PDE* 2015)

Let u be Lipschitz, semiconcave and such that $\mathcal{D}_s u \geq \eta_0 > 0$ in a ball B.

Fractional MA equation

Caffarelli and Charro considered the Dirichlet problem

$$\begin{cases} \mathcal{D}_s \bar{u} = \bar{u} - \phi & \text{in } \mathbb{R}^n \\ \lim_{|x|\to\infty} (\bar{u} - \phi)(x) = 0 \end{cases}$$

where ϕ is convex and behaves like a cone at infinity.

In particular, $\mathcal{D}_s \phi \geq 0 = \phi - \phi$, so ϕ is a subsolution and $\bar{u} \geq \phi$.

There exists a unique viscosity solution $\bar{u}$ that is **Lipschitz** and **semiconcave**.

Moreover, $\bar{u}$ has the crucial property

$$\bar{u} > \phi \quad \text{in } \mathbb{R}^n$$

Theorem (Caffarelli–Charro, *Ann. of PDE* 2015)

Let u be Lipschitz, semiconcave and such that $\mathcal{D}_s u \geq \eta_0 > 0$ in a ball B. Then **the equation becomes uniformly elliptic**: *there exists $\lambda > 0$ such that*

$$\mathcal{D}_s u(x) = \mathcal{D}_s^\lambda u(x) = \inf_{A > \lambda I, \det(A) = 1} \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x+z) + u(x-z) - 2u(x)}{|A^{-1} z|^{n+2s}} \, dz$$

Fractional MA equation

Caffarelli and Charro considered the Dirichlet problem

$$\begin{cases} \mathcal{D}_s \bar{u} = \bar{u} - \phi & \text{in } \mathbb{R}^n \\ \lim_{|x|\to\infty} (\bar{u} - \phi)(x) = 0 \end{cases}$$

where ϕ is convex and behaves like a cone at infinity.

In particular, $\mathcal{D}_s \phi \geq 0 = \phi - \phi$, so ϕ is a subsolution and $\bar{u} \geq \phi$.

There exists a unique viscosity solution $\bar{u}$ that is **Lipschitz** and **semiconcave**.

Moreover, $\bar{u}$ has the crucial property

$$\bar{u} > \phi \quad \text{in } \mathbb{R}^n$$

Theorem (Caffarelli–Charro, *Ann. of PDE* 2015)

Let u be Lipschitz, semiconcave and such that $\mathcal{D}_s u \geq \eta_0 > 0$ in a ball B.
Then **the equation becomes uniformly elliptic**: *there exists $\lambda > 0$ such that*

$$\mathcal{D}_s u(x) = \mathcal{D}_s^\lambda u(x) = \inf_{A > \lambda I, \det(A)=1} \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x+z) + u(x-z) - 2u(x)}{|A^{-1}z|^{n+2s}}\, dz$$

- The uniformly elliptic regularity theory of Caffarelli–Silvestre applies.

Obstacle problem for fractional MA equation

We consider the obstacle problem

$$\begin{cases} \mathcal{D}_s u \geq u - \phi & \text{in } \mathbb{R}^n \\ u \leq \psi & \text{in } \mathbb{R}^n \\ \mathcal{D}_s u = u - \phi & \text{in } \{u < \psi\} \\ \lim_{|x|\to\infty}(u-\phi)(x) = 0 \end{cases}$$

for an obstacle ψ such that

$$\psi > \phi \quad \text{and} \quad \psi \leq \bar{u} \ \text{ in some compact set.}$$

Obstacle problem for fractional MA equation

We consider the obstacle problem

$$\begin{cases} \mathcal{D}_s u \geq u - \phi & \text{in } \mathbb{R}^n \\ u \leq \psi & \text{in } \mathbb{R}^n \\ \mathcal{D}_s u = u - \phi & \text{in } \{u < \psi\} \\ \lim_{|x|\to\infty}(u-\phi)(x) = 0 \end{cases}$$

for an obstacle ψ such that

$$\psi > \phi \quad \text{and} \quad \psi \leq \bar{u} \quad \text{in some compact set.}$$

Theorem (Jhaveri–Stinga, 2017)

There exists a unique viscosity solution u that is Lipschitz and semiconcave **with constants depending only on ϕ and ψ.** *Moreover*

$$u > \phi \quad \textit{in } \mathbb{R}^n$$

Higher regularity of u and regularity of the free boundary $\partial\{u < \psi\}$.

Obstacle problem for fractional MA equation

We consider the obstacle problem

$$\begin{cases} \mathcal{D}_s u \geq u - \phi & \text{in } \mathbb{R}^n \\ u \leq \psi & \text{in } \mathbb{R}^n \\ \mathcal{D}_s u = u - \phi & \text{in } \{u < \psi\} \\ \lim_{|x|\to\infty}(u-\phi)(x) = 0 \end{cases}$$

for an obstacle ψ such that

$$\psi > \phi \quad \text{and} \quad \psi \leq \bar{u} \quad \text{in some compact set.}$$

Theorem (Jhaveri–Stinga, 2017)

There exists a unique viscosity solution u that is Lipschitz and semiconcave **with constants depending only on ϕ and ψ.** *Moreover*

$$u > \phi \quad \textit{in } \mathbb{R}^n$$

Higher regularity of u and regularity of the free boundary $\partial\{u < \psi\}$.

In particular, **locally**, the operator becomes uniformly elliptic.

Existence and regularity

- **Existence.** Very delicate due to degeneracy

Existence and regularity

- **Existence.** Very delicate due to degeneracy
- **Regularity.** Given any ball B, there exists $\lambda > 0$ such that

$$\begin{cases} \mathcal{D}_s^\lambda u \geq u - \phi & \text{in } \mathbb{R}^n \\ u \leq \psi & \text{in } \mathbb{R}^n \\ \mathcal{D}_s^\lambda u = u - \phi & \text{in } \{u < \psi\} \cap B \\ \lim_{|x|\to\infty}(u-\phi)(x) = 0 \end{cases}$$

Existence and regularity

- **Existence.** Very delicate due to degeneracy
- **Regularity.** Given any ball B, there exists $\lambda > 0$ such that

$$\begin{cases} \mathcal{D}_s^\lambda u \geq u - \phi & \text{in } \mathbb{R}^n \\ u \leq \psi & \text{in } \mathbb{R}^n \\ \mathcal{D}_s^\lambda u = u - \phi & \text{in } \{u < \psi\} \cap B \\ \lim_{|x|\to\infty}(u-\phi)(x) = 0 \end{cases}$$

This is a uniformly elliptic obstacle problem as considered in

- Caffarelli–Ros-Oton–Serra, Obstacle problems for integro-differential operators: regularity of solutions and free boundaries, *Invent. Math.* (2017).

Existence and regularity

- **Existence.** Very delicate due to degeneracy
- **Regularity.** Given any ball B, there exists $\lambda > 0$ such that

$$\begin{cases} \mathcal{D}_s^\lambda u \geq u - \phi & \text{in } \mathbb{R}^n \\ u \leq \psi & \text{in } \mathbb{R}^n \\ \mathcal{D}_s^\lambda u = u - \phi & \text{in } \{u < \psi\} \cap B \\ \lim_{|x|\to\infty}(u-\phi)(x) = 0 \end{cases}$$

This is a uniformly elliptic obstacle problem as considered in

- Caffarelli–Ros-Oton–Serra, Obstacle problems for integro-differential operators: regularity of solutions and free boundaries, *Invent. Math.* (2017).

We have **no equation** in the part of $\{u < \psi\}$ that is outside of B.

Existence and regularity

- **Existence.** Very delicate due to degeneracy
- **Regularity.** Given any ball B, there exists $\lambda > 0$ such that

$$\begin{cases} \mathcal{D}_s^\lambda u \geq u - \phi & \text{in } \mathbb{R}^n \\ u \leq \psi & \text{in } \mathbb{R}^n \\ \mathcal{D}_s^\lambda u = u - \phi & \text{in } \{u < \psi\} \cap B \\ \lim_{|x|\to\infty}(u-\phi)(x) = 0 \end{cases}$$

This is a uniformly elliptic obstacle problem as considered in

- Caffarelli–Ros-Oton–Serra, Obstacle problems for integro-differential operators: regularity of solutions and free boundaries, *Invent. Math.* (2017).

We have **no equation** in the part of $\{u < \psi\}$ that is outside of B.

Careful. Nonlocal information may have local effects: Dipierro–Savin–Valdinoci, All functions are locally s-harmonic up to a small error, *JEMS* (2017).

Existence and regularity

- **Existence.** Very delicate due to degeneracy
- **Regularity.** Given any ball B, there exists $\lambda > 0$ such that

$$\begin{cases} \mathcal{D}_s^\lambda u \geq u - \phi & \text{in } \mathbb{R}^n \\ u \leq \psi & \text{in } \mathbb{R}^n \\ \mathcal{D}_s^\lambda u = u - \phi & \text{in } \{u < \psi\} \cap B \\ \lim_{|x|\to\infty} (u - \phi)(x) = 0 \end{cases}$$

This is a uniformly elliptic obstacle problem as considered in

- Caffarelli–Ros-Oton–Serra, Obstacle problems for integro-differential operators: regularity of solutions and free boundaries, *Invent. Math.* (2017).

We have **no equation** in the part of $\{u < \psi\}$ that is outside of B.

Careful. Nonlocal information may have local effects: Dipierro–Savin–Valdinoci, All functions are locally s-harmonic up to a small error, *JEMS* (2017).

We are good. The global control $\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \leq 1$ permits us to obtain the same blow ups near free boundary points as in Caffarelli–Ros-Oton–Serra.

Thank you for your attention!