Obstacle problems for nonlocal operators

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Outline

Motivation

Optimal regularity of solutions

Regularity of the free boundary

Selected references

Motivation I

- Pricing of (perpetual) American options when the underlying asset price is a pure-jump Markov process.

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- The asset price $\{S(t)\}_{t\geq 0}$ is characterized by the infinitesimal generator:

$$Au(x) = \int_{\mathbb{R}^n \setminus \{0\}} \left(u(x+y) - u(x) - y \cdot \nabla u(x) \mathbf{1}_{\{|y|\leq 1\}} \right) d\nu(y) \\ + b(x) \cdot \nabla u(x),$$

where $d\nu(y)$ is a Lévy measure.

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  where $d\nu(y)$ is a Lévy measure.
- Consider a perpetual American option written on the underlying $\{S(t)\}_{t\geq 0}$ with payoff $\varphi(x)$.

Motivation II

- We assume that the perpetual American option prices is given by

$$u(x) := \sup_{\tau \in \mathscr{T}} \mathbb{E}\left[e^{-r\tau} \varphi(S(\tau)) \middle| S(0) = x \right],$$

where the asset price process $\{S(t)\}_{t \geq 0}$ is specified under a suitable probability measure.

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- We expect $u(x)$ to solve the system of complementarity conditions:

$$\begin{aligned} u \geq \varphi \quad & \text{on } \mathbb{R}^n, \\ -Au + ru = 0 \quad & \text{on } \{u > \varphi\}, \\ -Au + ru \geq 0 \quad & \text{on } \{u = \varphi\}, \end{aligned}$$

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  or, more compactly, we have that

$$\min\{-Au(x) + ru(x), u(x) - \varphi(x)\} = 0, \quad \forall\, x \in \mathbb{R}^n.$$

Main questions

For the stationary obstacle problem,

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We will present results about the previous two questions in the case when the nonlocal operator A is the fractional Laplacian with drift, that is,

$$Au(x) = -(-\Delta)^s u(x) + b(x) \cdot \nabla u(x), \quad \forall x \in \mathbb{R}^n,$$

where $s \in (0, 1)$.

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Models for asset prices related to our research can be written as a subordinated Brownian motion:

- Normal Inverse Gaussian processes (Barndorff-Nielsen (1997-1998));
- Variance Gamma processes (Madan and Seneta (1990));
- Tempered stable processes (Koponen (1995), Boyarchenko and Levendorskiĭ (2000), Carr, Geman, Madan, and Yor (2002-2003)).

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- The process $X(t) := Z(T(t))$ is called a Normal Inverse Gaussian process and is characterized by the Lévy measure,

$$\nu(x) = \frac{C}{|x|} e^{Ax} K_1(B|x|),$$

where $A = \theta$, $B = \sqrt{\theta^2 + 1}$, $C = B/(2\pi)$, and $K_1(z)$ is the modified Bessel function of the second kind.

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- The infinitesimal generator of $X(t)$ is

$$\begin{aligned} Au(x) &= \int_{\mathbb{R}} (u(x+y) - u(x)) d\nu(y) \\ &= 1 - \left(-\Delta u - 2\theta \cdot \nabla u + 1\right)^{1/2}(x). \end{aligned}$$

Inverse Gaussian subordinator

- The subordinator of the Normal Inverse Gaussian process can be written as the inverse local time of a one-dimensional Brownian motion with drift, with infinitesimal generator,

$$Lu(y) = \frac{1}{2}\frac{d^2u(y)}{dy^2} + \frac{du(y)}{dy}, \quad \forall\, y > 0.$$

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- We can write L as a Sturm-Liouville operator in the form,

$$Lu(y) = \frac{1}{2m(y)}\frac{d}{dy}\left(m(y)\frac{du}{dy}\right)(y), \quad \forall\, y > 0,$$

where we used the weight function,

$$m(y) = 2e^{2y}.$$

Dirichlet-to-Neumann map

- This implies that the generator of the Normal Inverse Gaussian process is the Dirichlet-to-Neumann map for the extension operator:

$$Ev(x, y) = \frac{1}{2} v_{xx} + \theta v_x + \frac{1}{2} v_{yy} + v_y,$$

for all $(x, y) \in \mathbb{R} \times (0, \infty)$.

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for all $(x,y) \in \mathbb{R} \times (0,\infty)$.

- In other words, we have that if $v \in C(\mathbb{R} \times [0,\infty))$ is a solution to the Dirichlet problem,

$$\begin{cases} Ev(x,y) = 0, & \forall\,(x,y) \in \mathbb{R} \times (0,\infty), \\ v(x,0) = v_0(x), & \forall\,x \in \mathbb{R}, \end{cases}$$

then we have that

$$\lim_{y \downarrow 0} m(y) v_y(x,y) = 2 \lim_{y \downarrow 0} v_y(x,y) = Av_0(x), \quad \forall\,x \in \mathbb{R}.$$

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- The process $X(t) := Z(T(t))$ is called a Variance Gamma process and is characterized by the Lévy measure,

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$$\begin{aligned} Au(x) &= \int_{\mathbb{R}} (u(x+y) - u(x)) d\nu(y) \\ &= -\log\left(-\frac{1}{2}\Delta u - \theta \cdot \nabla u + 1\right)(x). \end{aligned}$$

Gamma subordinator

- Donati-Martin and Yor (2005) prove that the subordinator of the Variance Gamma process can be written as the inverse local time of a one-dimensional diffusion process with infinitesimal generator,

$$Lu(y) = \frac{1}{2}\frac{d^2u(y)}{dy^2} + \left(\frac{1}{2y} + \sqrt{2}\frac{K_0'(\sqrt{2}y)}{K_0(\sqrt{2}y)}\right)\frac{du(y)}{dy}, \quad \forall\, y > 0,$$

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  then we have that

$$\lim_{y \downarrow 0} m(y) v_y(x,y) = A v_0(x), \quad \forall x \in \mathbb{R}.$$

Tempered stable processes

- A similar analysis can be done for the class of tempered stable processes, which are (roughly) characterized by the Lévy measure,

$$\nu(x) = \frac{C}{|x|^{1+\alpha}} e^{Ax - B|x|},$$

where A, B, C are positive constants, $A < B$, and $\alpha \in (0, 2)$.

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- To our knowledge, it is not known a closed form expression for a one-dimensional diffusion whose inverse local time at the origin is equal to the subordinator of the tempered stable process.

- Necessary and sufficient conditions for subordinators that can be written as inverse local time of generalized diffusions were obtained by Knight (1981), and Kotani and Watanabe (1982).

Obstacle problems for nonlocal operators

- Up to not long ago, viewing the nonlocal operator as a Dirichlet-to-Neumann map (or, equivalently, the underlying Lévy process as a subordinated Brownian motion, where the subordinator is the inverse local time of a one-dimensional diffusion) was the unique method to analyze obstacle problems for nonlocal operators.

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- Caffarelli, Ros-Oton, and Serra (2016) develop a new method that applies to all homogeneous Lévy measures that are symmetric about the origin, and does not use the previous property.
- The above mentioned models used in mathematical finance do not in general satisfy the assumptions in the Caffarelli, Ros-Oton, and Serra (2016) article.

Symmetric $2s$-stable processes

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- The generator of symmetric 2s-stable process can be represented in integral form as

$$Au(x) = \int_{\mathbb{R}^n} \left(u(x+y) - u(x) - y \cdot \nabla u(x) \mathbf{1}_{\{|y|<1\}} \right) \frac{1}{|y|^{n+2s}},$$

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where $s \in (0,1)$.

- Using a functional-analytic framework, we can also represent A as

$$Au = -(-\Delta)^s u.$$

Symmetric stable processes with drift

- We consider a generalization of symmetric stable processes by adding a drift component, that is, we study operators of the form

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$$Au(x) = -(-\Delta)^s u(x) + b(x) \cdot \nabla u(x) + c(x)u(x), \quad \forall x \in \mathbb{R}^n.$$

- The strength of the gradient perturbation is most easily seen in the Fourier variables:

$$-Au(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(|\xi|^{2s} + ib(x)\cdot\xi + c(x)\right) \widehat{u}(\xi), \quad \forall\, \xi \in \mathbb{R}^n.$$

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$$a(x,\xi) = |\xi|^{2s} + ib(x) \cdot \xi + c(x), \quad \forall\, x, \xi \in \mathbb{R}^n.$$

- The properties of the symbol, $a(x,\xi)$, change depending on whether

$$2s < 1, \qquad 2s = 1, \qquad \text{or} \qquad 2s > 1.$$

Properties of the symbol

$$a(x,\xi) = |\xi|^{2s} + ib(x)\cdot\xi + c(x), \quad \forall\, x,\xi \in \mathbb{R}^n.$$

We have three cases:

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2. If $2s = 1$ (critical regime): the jump and drift component in $a(x,\xi)$ have the same contribution, but they imply different regularity properties.

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1. If $2s < 1$ (supercritical regime): the drift component in $a(x,\xi)$ has the strongest contribution and the operator is not elliptic, so standard theory does not apply.
2. If $2s = 1$ (critical regime): the jump and drift component in $a(x,\xi)$ have the same contribution, but they imply different regularity properties.
3. If $2s > 1$ (subcritical regime): the jump component in $a(x,\xi)$ has the strongest contribution, which makes the operator elliptic, and so we expect the standard properties of elliptic operators to hold.

Obstacle problem

When $2s > 1$, we study the stationary obstacle problem defined by the fractional Laplacian with drift,

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and we prove:

- Existence, uniqueness, and optimal regularity C^{1+s} of solutions;
- The $C^{1+\gamma}$ regularity of the regular part of the free boundary.

Optimal regularity of solutions

Existence and optimal regularity of solutions

Theorem (Petrosyan-P.)

Let $1 < 2s < 2$*.*

Assume that $b \in C^s(\mathbb{R}^n;\mathbb{R}^n)$*, and* $c \in C^s(\mathbb{R}^n)$ *is a nonnegative function. Assume that the obstacle function,* $\varphi \in C^{3s}(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$*, is such that*

$$(A\varphi)^+ \in L^\infty(\mathbb{R}^n).$$

Then there is a solution, $u \in C^{1+s}(\mathbb{R}^n)$*, to the obstacle problem defined by the fractional Laplacian with drift.*

Uniqueness of solutions

Theorem (Petrosyan-P.)

Let $0 < 2s < 2$ *and* $\alpha \in ((2s-1) \vee 0, 1)$.

Assume that $b \in C(\mathbb{R}^n; \mathbb{R}^n)$ *is a Lipschitz continuous function, and* $c \in C(\mathbb{R}^n)$ *is such that there is a positive constant,* c_0, *such that*

$$c(x) \geq c_0 > 0, \quad \forall x \in \mathbb{R}^n.$$

Assume that the obstacle function, $\varphi \in C(\mathbb{R}^n)$.

Then there is at most one solution, $u \in C^{1+\alpha}(\mathbb{R}^n)$, *to the obstacle problem defined by the fractional Laplacian with drift.*

Stochastic representations of solutions

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- Uniqueness of solutions is established by proving their stochastic representation.
- Let $(\Omega, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space, and let $N(dt, dx)$ be a Poisson random measure with Lévy measure,

$$d\nu(x) = \frac{dx}{|x|^{n+2s}},$$

and let $\widetilde{N}(dt, dx)$ be the compensated Poisson random measure.

Stochastic representations of solutions

- Uniqueness of solutions is established by proving their stochastic representation.
- Let $(\Omega, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space, and let $N(dt, dx)$ be a Poisson random measure with Lévy measure,

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- Let $\{X(t)\}_{t\geq 0}$ be the unique RCLL solution to the stochastic equation,

$$X(t) = X(0) + \int_0^t b(X(s-))\, ds + \int_0^t \int_{\mathbb{R}^n\setminus\{O\}} x\widetilde{N}(ds, dx), \quad \forall t > 0.$$

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- Then, if $u \in C^{1+\alpha}(\mathbb{R}^n)$ is a solution to the obstacle problem, for some $\alpha \in ((2s-1) \vee 0, 1)$, we have that

$$u(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^x \left[e^{-\int_0^\tau c(X(s-))\, ds} \varphi(X(\tau)) \right], \quad \forall x \in \mathbb{R}^n,$$

where $\mathcal{T}$ denotes the set of stopping times.

Remarks on uniqueness

- The Lipschitz continuity of the vector field $b(x)$ is used to ensure the existence and uniqueness of solutions, $\{X(t)\}_{t \geq 0}$, to the stochastic equation.

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- The condition that the zeroth order coefficient, $c(x) \geq c_0 > 0$, is used to ensure that the expression on the right-hand side of the stochastic representation is finite even for unbounded stopping times, τ.
- If $\{X(t)\}_{t\geq 0}$ were an asset price process, and the law of the process were a risk-neutral probability measure, then the stochastic representation indicates that u is the value function of an perpetual American option with payoff φ on the underlying $\{X(t)\}_{t\geq 0}$.

Optimal regularity of solutions

- The optimal regularity of solutions to the obstacle problem for the fractional Laplace operator without drift was studied by Caffarelli-Salsa-Silvestre (2008), under the assumption that the obstacle function, $\varphi \in C^{2,1}(\mathbb{R}^n)$, and by Silvestre (2007), under the assumption that the contact set $\{u = \varphi\}$ is convex.

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- To obtain the optimal regularity of solutions, we reduce our problem to an obstacle problem without drift,

$$\min\{(-\Delta)^s \tilde{u}, \tilde{u} - \tilde{\varphi}\} = 0 \quad \text{on } \mathbb{R}^n,$$

  for which we can at most assume that $\tilde{\varphi} \in C^{2s+\alpha}(\mathbb{R}^n)$, for all $\alpha \in (0, s)$.

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  for which we can at most assume that $\tilde{\varphi} \in C^{2s+\alpha}(\mathbb{R}^n)$, for all $\alpha \in (0, s)$.
- From now on we consider the reduced problem and we write u instead of $\tilde{u}$ and φ instead of $\tilde{\varphi}$.

Extension operator

- For $s \in (0,1)$, let $a = 1 - 2s$ and consider the degenerate-elliptic operator,

$$L_a v = \frac{1}{2}\Delta v + \frac{1-2s}{2y}\frac{\partial v}{\partial y},$$

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which can be written in divergence form as

$$L_a v(x,y) = \frac{1}{2m(y)}\operatorname{div}\left(m(y)\nabla v\right)(x,y), \quad \forall\,(x,y) \in \mathbb{R}^n \times \mathbb{R}_+,$$

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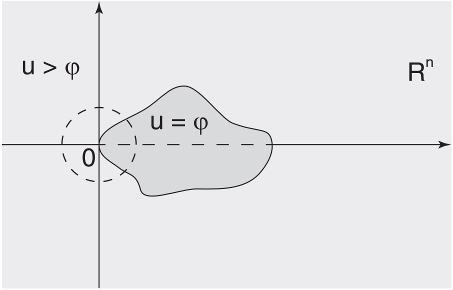
where $m(y) = y^a$.
- Molchanov-Ostrovskii (1969) and Caffarelli-Silvestre (2007) prove that, if v is a L_a-harmonic function such that
$$\begin{cases} L_a v(x,y) &= 0, & \forall\,(x,y) \in \mathbb{R}^n \times (0,\infty), \\ v(x,0) &= v_0(x), & \forall\, x \in \mathbb{R}^n, \end{cases}$$
then we have that
$$\lim_{y \downarrow 0} m(y) v_y(x,y) = -(-\Delta)^s v_0(x), \quad \forall\, x \in \mathbb{R}^n.$$

Steps to prove the optimal regularity of solutions

- We only need to study the regularity of the solutions in a neighborhood of free boundary points:



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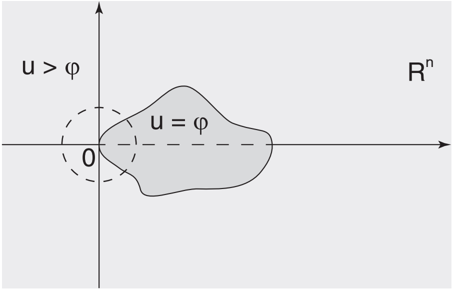

- We consider the height function

$$v(x) := u(x) - \varphi(x),$$

and the goal is to establish the growth estimate:

$$0 \leq v(x) \leq C|x|^{1+s}.$$

Steps to prove the optimal regularity of solutions I

- Let $u(x, y)$ and $\varphi(x, y)$ be the L_a-harmonic extensions and let:

  $$v(x, y) := u(x, y) - \varphi(x, y) + (-\Delta)^s \varphi(O) |y|^{1-a}, \quad \forall\, (x, y) \in \mathbb{R}^n \times \mathbb{R}_+.$$

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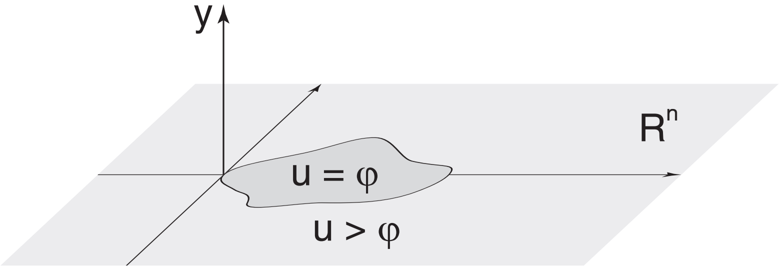
- Extend v by even symmetry across $\{y = 0\}$.
- The height function $v(x,y)$ satisfies the following conditions:

$$\begin{aligned} L_a v &= 0 && \text{on } \mathbb{R}^n \times (\mathbb{R}\backslash\{0\}), \\ L_a v(x,y) &\leq h(x)\mathcal{H}^n|_{\{y=0\}} && \text{on } \mathbb{R}^{n+1}, \\ L_a v(x,y) &= h(x)\mathcal{H}^n|_{\{y=0\}} && \text{on } \mathbb{R}^{n+1}\backslash\{u = \varphi\}, \end{aligned}$$



Steps to prove the optimal regularity of solutions II

We need a suitable monotonicity formula of Almgren-type to find the lowest degree of regularity of the solution.

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Theorem (Almgren (1979))

Let u be a harmonic function. Then the function

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is non-decreasing in $r \in (0,1)$.

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is non-decreasing in $r \in (0, 1)$.

Moreover, $\Phi_u(r)$ is constant if and only if $\Phi_u(r) = k$, for some $k = 0, 1, 2, \ldots$, and u is a homogeneous harmonic function of degree k.

Steps to prove the optimal regularity of solutions III

- We will establish a version of the monotonicity formula for the function:

$$\Phi_v^p(r) := r\frac{d}{dr}\log\max\left\{\int_{\partial B_r}|v|^2|y|^{1-2s}, r^{n+1-2s+2(1+p)}\right\},$$

where r and p are positive constants.

Steps to prove the optimal regularity of solutions III

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  where r and p are positive constants.

- To see the connection with Almgren's classical monotonicity formula, omitting some technical details, the function $\Phi_v^p(r)$ takes the form:

$$\Phi_v^p(r) := 2r\frac{\int_{B_r}|\nabla v|^2|y|^{1-2s}}{\int_{\partial B_r}v^2|y|^{1-2s}} + (n+1-2s) + \text{“some noise”}.$$

Steps to prove the optimal regularity of solutions IV

Theorem (Almgren-type monotonicity formula)

Let $s \in (1/2, 1)$, $\alpha \in (1/2, s)$ and $p \in [s, \alpha + s - 1/2)$. Then there are positive constants, C and γ, such that the function

$$r \mapsto e^{Cr^{\gamma}} \Phi_v^p(r)$$

is non-decreasing, and we have that

$$\Phi_v(0+) \geq 2(1+s) + (n+1-2s).$$

Steps to prove the optimal regularity of solutions IV

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Remark

Omitting some technical conditions, the lower bound

$$\Phi_v(0+) \geq 2(1+s) + (n+1-2s)$$

allows us to prove that the limit of the sequence of Almgren-type rescalings $\{v_r\}$, as $r \downarrow 0$, is a homogeneous function of degree at least $1+s$.

Steps to prove the optimal regularity of solutions V

We study the properties of the sequence of Almgren-type rescalings:

$$v_r(x,y) := \frac{v(r(x,y))}{d_r}, \text{ where } d_r := \left(\frac{1}{r^{n+a}}\int_{\partial B_r} |v|^2|y|^a\right)^{1/2}.$$

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Lemma (Uniform Schauder estimates)

Let $\alpha \in ((2s-1)\vee 1/2, s)$ *and* $p \in [s, \alpha+s-1/2)$.

Assume that $u \in C^{1+\alpha}(\mathbb{R}^n)$ *and* $\varphi \in C^{2s+\alpha}(\mathbb{R}^n)$, *and that* $\liminf_{r\to 0}\frac{d_r}{r^{1+p}} = \infty$.

Then there are positive constants, C, $\gamma \in (0,1)$ *and* r_0, *such that*

$$\begin{aligned} \|v_r\|_{C^\gamma(\bar{B}^+_{1/8})} &\leq C, \\ \|\partial_{x_i} v_r\|_{C^\gamma(\bar{B}^+_{1/8})} &\leq C, \quad \forall i = 1,\dots,n, \\ \||y|^a\partial_y v_r\|_{C^\gamma(\bar{B}^+_{1/8})} &\leq C, \end{aligned}$$

for all $r \in (0, r_0)$.

Steps to prove the optimal regularity of solutions VI

- Almgren monotonicity formula and the compactness of the sequence of rescalings imply the growth estimate

$$0 \leq v(x) \leq C|x|^{1+s}, \quad \forall x \in B_{r_0}(O).$$

Steps to prove the optimal regularity of solutions VI

- Almgren monotonicity formula and the compactness of the sequence of rescalings imply the growth estimate

$$0 \leq v(x) \leq C|x|^{1+s}, \quad \forall x \in B_{r_0}(O).$$

- Optimal regularity, that is, $v \in C^{1+s}(\mathbb{R}^n)$, is a consequence of the preceding growth estimate of u.

Regularity of the free boundary

Regular free boundary points

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Theorem (Garofalo-Petrosyan-P.-Smit)

The regular free boundary, $\Gamma_{1+s}(u)$, is a relatively open set and is locally $C^{1+\gamma}$, for a constant $\gamma = \gamma(n, s) \in (0, 1)$.

Comparison with previous research

- The $C^{1+\gamma}$ regularity of the regular free boundary was obtained by Caffarelli-Salsa-Silvestre (2008) for the fractional Laplacian without drift in the case when the obstacle function $\varphi \in C^{2,1}(\mathbb{R}^n)$.

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- This approach does not have an obvious generalization to the case when the obstacle function has a lower degree of monotonicity, that is, $\varphi \in C^{2s+\alpha}(\mathbb{R}^n)$, for all $\alpha \in (0, s)$.
- Instead we adapt Weiss' approach (1998) of the proof of the regularity of the regular free boundary from the case of the Laplace operator to that of the fractional Laplacian, which in addition allows us to work with lower degree of regularity of the obstacle function.

Main idea of the proof I

We fix a regular free boundary point $x_0 \in \Gamma_{1+s}$.

- Because we know the optimal regularity of solutions, we can now consider the homogeneous rescalings:

$$v_{x_0,r}(x,y) := \frac{1}{r^{1+s}} v(x_0 + rx, ry), \quad \forall\,(x,y) \in \mathbb{R}^n \times \mathbb{R}.$$

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- The homogeneous rescalings converge to a non-trivial homogeneous solution in the class of functions:

$$\mathcal{H}_{1+s} := \Big\{ a \left(x \cdot e + \sqrt{(x\cdot e)^2 + y^2} \right)^s \left(x \cdot e - s\sqrt{(x\cdot e)^2 + y^2} \right) : \\ a > 0,\, e \in \mathbb{R}^n,\, |e| = 1 \Big\}.$$

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Theorem (Garofalo-Petrosyan-P.-Smit)

Let $x_0 \in \Gamma_{1+s}(u)$. Then there are positive constants C, η and $\gamma = \gamma(n, s)$, such that for all $x', x'' \in \Gamma \cap B_\eta(x_0)$, we have that

$$|a_{x'} - a_{x''}| \leq C|x' - x''|^\gamma,$$
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- The $C^{1+\gamma}$-regularity of the regular free boundary $\Gamma_{1+s}(u)$ is a direct consequence of the previous estimates.
- The previous estimates are a consequence of a version of a Weiss monotonicity formula and an epiperimetric inequality adapted to the framework of the fractional Laplacian.

Weiss-type monotonicity formula

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- We define the Weiss-type functional by letting:

$$
\begin{aligned}
W_L(v,r,x_0) &:= \frac{1}{r^{n+2}} I_{x_0}(r) - \frac{1+s}{r^{n+3}} F_{x_0}(r), \\
I_{x_0}(r) &:= \int_{B_r(x_0)} |\nabla v_{x_0}|^2 |y|^{1-2s} + \int_{B_r'(x_0)} v_{x_0} h_{x_0}, \\
F_{x_0}(r) &:= \int_{\partial B_r(x_0)} |v_{x_0}|^2 |y|^{1-2s},
\end{aligned}
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where $B_r' = B_r \cap \{y = 0\}$.

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$$F_{x_0}(r) := \int_{\partial B_r(x_0)} |v_{x_0}|^2 |y|^{1-2s},$$

where $B'_r = B_r \cap \{y = 0\}$.

Theorem (Monotonicity of the Weiss functional)

There are constants $C, r_0 > 0$ such that for all $x_0 \in \Gamma(u)$ we have that

$$r \mapsto W_L(v, r, x_0) + Cr^{2s-1}$$

is nondecreasing on $(0, r_0)$.

Epiperimetric inequality

We define the boundary adjusted Weiss energy by letting:

$$W(v) := \int_{B_1} |\nabla v|^2 |y|^{1-2s} - (1+s) \int_{\partial B_1} v^2 |y|^{1-2s}.$$

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Theorem (Epiperimetric inequality)

There are constants $\kappa, \delta \in (0,1)$ *such that if* $w \in H^1(B_1, |y|^{1-2s})$ *is a homogeneous function of degree* $(1+s)$ *such that*

$$\begin{aligned} &w \geq 0 \quad \text{on } B_1 \cap \{y = 0\}, \\ &dist(w, \mathcal{H}_{1+s}) < \delta, \end{aligned}$$

then there is $\widehat{w} \in H^1(B_1, |y|^{1-2s})$ *such that*

$$\begin{aligned} &\widehat{w} \geq 0 \quad \text{on } B_1 \cap \{y = 0\}, \\ &\widehat{w} = w \quad \text{on } \partial B_1, \end{aligned}$$

and we have that $\qquad W(\widehat{w}) \leq (1-\kappa) W(w).$

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Conclusions

- In the analysis of the obstacle problem for the fractional Laplacian with drift, it was essential to know that the fractional Laplacian operator can be viewed as the Dirichlet-to-Neumann map for a local extension operator.
- This allowed us to use local methods to adapt the concepts of monotonicity formulas already developed for model local operators to the framework of nonlocal operators.
- This is a property shared by many models important in financial engineering, such as the generators of the Normal Inverse Gaussian process, Variance Gamma process, and Tempered stable process.

Conclusions

- In the analysis of the obstacle problem for the fractional Laplacian with drift, it was essential to know that the fractional Laplacian operator can be viewed as the Dirichlet-to-Neumann map for a local extension operator.
- This allowed us to use local methods to adapt the concepts of monotonicity formulas already developed for model local operators to the framework of nonlocal operators.
- This is a property shared by many models important in financial engineering, such as the generators of the Normal Inverse Gaussian process, Variance Gamma process, and Tempered stable process.
- In the future, we hope to extend these methods to the study of the obstacle problem associated to the previously mentioned processes and their lower order perturbations.

THANK YOU!

Bibliography I

D. Applebaum
Lévy processes and stochastic calculus
Cambridge Studies in Advanced Mathematics (2004)

L. A. Caffarelli, X. Ros-Oton, J. Serra
Obstacle problems for integro-differential operators: Regularity of solutions and free boundaries
http://arxiv.org/pdf/1601.05843.pdf (2016)

L. A. Caffarelli, S. Salsa, L. Silvestre
Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian
Invent. Math. 171 (2008)

L. A. Caffarelli, L. Silvestre
An extension problem related to the fractional Laplacian
Comm. P.D.E. 32 (2007)

Bibliography II

P. Carr, H. Geman, D. B. Madan, M. Yor
Stochastic Volatility for Lévy Processes
Mathematical Finance 13 (2003)

R. Cont, P. Tankov
Financial modeling with jump processes
Chapman & Hall (2004)

C. L. Epstein, C. A. Pop
Regularity for the supercritical fractional Laplacian with drift
Journal of Geometric Analysis 26 (2) (2016)
http://arxiv.org/pdf/1309.5892.pdf

A. Petrosyan, C. A. Pop
Optimal regularity of solutions to the obstacle problem for the fractional Laplacian with drift
Journal of Functional Analysis 268 (2015), no. 2
http://arxiv.org/pdf/1403.5015.pdf

Bibliography III

N. Garofalo, A. Petrosyan, C. A. Pop, M. Smit Vega Garcia
Regularity of the free boundary for the obstacle problem for the fractional Laplacian with drift
accepted in Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire
http://arxiv.org/pdf/1509.06228.pdf

M. E. Taylor
Partial Differential Equations I. Basic Theory
Springer (2011)

M. E. Taylor
Partial Differential Equations II. Qualitative studies of linear equations
Springer (2011)

L. Silvestre
Regularity of the obstacle problem for a fractional power of the Laplace operator
Comm. Pure Appl. Math. 60 (2007)