BDDC Domain Decomposition Algorithms

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Problems considered

- BDDC stands for Balancing Domain Decomposition by Constraints and this family of algorithms was introduced by Clark Dohrmann in 2003 following the introduction of the FETI–DP (Dual Primal Finite Element Tearing and Interconnecting) algorithms by Charbel Farhat et al in 2000. Dohrmann remains a main provider of ideas and analysis.

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- In this talk, I will introduce the basics of BDDC, give examples of successful applications and talk about some recent work.
- My last few PhD students were all involved in the development of the BDDC family. I have also worked with Dohrmann and with Beirão da Veiga, Pavarino, Scacchi, Zampini, Oh, and Calvo. Recently, the focus has been on small coarse problems for BDDC, adaptive choices of the coarse problems, and isogeometric analysis problems for elasticity including the almost incompressible case..

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- All this aims at developing preconditioners for the stiffness matrices. These approximate inverses are then combined with conjugate gradients or other Krylov space methods.
- Primarily interested in hard problems with very many subdomains and having convergence rates independent of that number and with rates that decrease slowly with the size of the subdomain problems. Many bounds independent of jumps in coefficients between subdomains.

BDDC, finite element meshes, and equivalence classes

- BDDC algorithms work on decompositions of the domain Ω of the elliptic problem into non-overlapping subdomains Ω_i, each often with tens of thousands of degrees of freedom. In between the subdomains the interface Γ. The local interface of Ω_i: $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$. Γ does not cut any elements.

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- The nodes on Γ are partitioned into equivalence classes of sets of indices of the local interfaces Γ_i to which they belong. For 3D and nodal finite elements, we have classes of face nodes, associated with two local interfaces, and classes of edge nodes and subdomain vertex nodes.

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- The nodes on Γ are partitioned into equivalence classes of sets of indices of the local interfaces Γ_i to which they belong. For 3D and nodal finite elements, we have classes of face nodes, associated with two local interfaces, and classes of edge nodes and subdomain vertex nodes.
- For $H(\mathbf{curl})$ and Nédélec (edge) elements, only equivalence classes of element edges on subdomain faces and on subdomain edges. For $H(\text{div})$ and Raviart-Thomas elements, only degrees of freedom for element faces.

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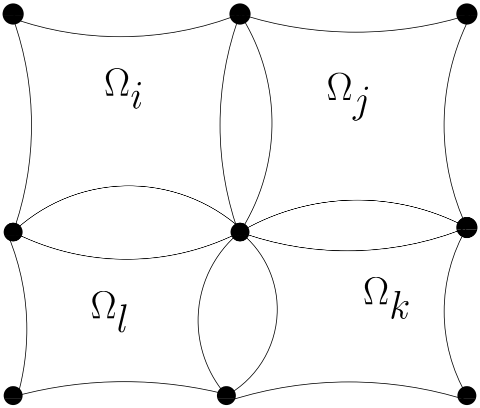
- When developing theory, we can now handle quite irregular subdomains, in particular, in 2D. In 3D, we obtain bounds in terms of the Lipschitz parameters of the subdomains, often obtained by a mesh partitioner.

Torn 2D scalar elliptic problem



Subdomain matrices

- Represent the subdomain stiffness matrix $A^{(i)}$ as

$$\begin{pmatrix} A^{(i)}_{II} & A^{(i)}_{I\Delta} & A^{(i)}_{I\Pi} \\ A^{(i)}_{\Delta I} & A^{(i)}_{\Delta\Delta} & A^{(i)}_{\Delta\Pi} \\ A^{(i)}_{\Pi I} & A^{(i)}_{\Pi\Delta} & A^{(i)}_{\Pi\Pi} \end{pmatrix}.$$

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- This matrix represents the energy contributed by Ω_i.
- We enforce continuity of the primal variables, as in the given finite element model, but allow multiple values of the dual variables when working with the partially subassembled model.

Partially subassembled matrix

- Maintain continuity of the primal variables at the vertices. Partially subassemble and mark with tilde:

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Delta}^{(1)} & & & & A_{\Pi I}^{(1)T} \\ A_{\Delta I}^{(1)} & A_{\Delta\Delta}^{(1)} & & & & A_{\Pi\Delta}^{(1)T} \\ & & \ddots & & & \vdots \\ & & & A_{II}^{(N)} & A_{I\Delta}^{(N)} & A_{\Pi I}^{(N)T} \\ & & & A_{\Delta I}^{(N)} & A_{\Delta\Delta}^{(N)} & A_{\Pi\Delta}^{(N)T} \\ A_{\Pi I}^{(1)} & A_{\Pi\Delta}^{(1)} & \cdots & A_{\Pi I}^{(N)} & A_{\Pi\Delta}^{(N)} & \tilde{A}_{\Pi\Pi} \end{bmatrix}$$

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- BDDC: After solving, enforce the continuity constraints by using an averaging operator E_D.
- FETI–DP: Uses Lagrange multipliers. A saddle point problem is then reduced to an equation for the Lagrange multipliers.

More on BDDC and FETI–DP

- The partially subassembled stiffness matrix of this alternative finite element model is used to define preconditioners; the resulting linear system is much cheaper to solve than the fully assembled system. Primal variables provide a global component of these preconditioners. Makes matrices invertible.

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- In a FETI–DP algorithm, the continuity at the edge nodes enforced by using Lagrange multipliers and the rate of convergence enhanced by also solving a Dirichlet problem on each subdomain in each iteration. The conjugate gradient algorithm is used to find accurate enough values of Lagrange multipliers. There are subtle scaling issues.

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- In a BDDC algorithm, continuity is instead restored in each iteration by computing a weighted average across the interface. Can lead to non-zero residuals at nodes next to Γ. If so, use a subdomain Dirichlet solve to eliminate them, in each iteration.

Alternative sets of primal constraints

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- Good numerical results in 2D but for competitive algorithms in 3D, certain averages (and first order moments) of the displacement over individual edges (and faces) should also take common values across interface Γ. Same matrix structure as before after a change of variables.
- Reliable recipes exist for selecting small sets of primal constraints for elasticity in 3D, which primarily use edge averages and first order moments as primal constraints. High quality PETSc-based codes have been developed by Stefano Zampini and been successfully tested on very large systems. Public domain software. Also great work by Klawonn's group.

Product spaces

- The BDDC and FETI–DP algorithms can be described in terms of three product spaces of finite element functions/vectors defined by their interface nodal values:

$$\widehat{W}_\Gamma \subset \widetilde{W}_\Gamma \subset W_\Gamma.$$

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- After eliminating the interior variables, write the subdomain Schur complements as

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- Partially subassemble the $S^{(i)}$, obtaining $\tilde{S}$ and a global problem.

More details on BDDC

- Work with $\widetilde{W}_\Gamma$ and a set of primal constraints. At the end of each iterative step, the approximate solution will be made continuous at all nodal points of the interface; continuity is restored by applying a weighted average operator E_D, which maps $\widetilde{W}_\Gamma$ into $\widehat{W}_\Gamma$.

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- In final step of each iteration, eliminate these residuals by solving a Dirichlet problem on each of the subdomains. Accelerate with preconditioned conjugate gradients.

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- E_D is a projection and $\widetilde{R}_{D\Gamma}^T \widetilde{R}_\Gamma = I$.

Condition number bounds

Lemma

(Lower bound)

$$u^T M_{BDDC} u \leq u^T \widetilde{S}_\Gamma u, \quad \forall u \in \widetilde{W}_\Gamma. \tag{1}$$

Proof.

Let $w = M_{BDDC} u$. Since $\widetilde{R}_\Gamma^T \widetilde{R}_{D,\Gamma} = I$, we have

$$\begin{aligned}
u^T M_{BDDC} u \leq u^T w &= u^T \widetilde{R}_\Gamma^T \widetilde{R}_{D,\Gamma} w = u^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w \\
&\leq (\widetilde{R}_\Gamma u, \widetilde{R}_\Gamma u)_{\widetilde{S}_\Gamma}^{1/2} (\widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w, \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w)_{\widetilde{S}_\Gamma}^{1/2} \\
&= (u^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma u)^{1/2} (w^T \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w)^{1/2} \\
&= (u^T \widetilde{S}_\Gamma u)^{1/2} (u^T M_{BDDC} u)^{1/2}.
\end{aligned}$$

Square and cancel common factor. ∎

Condition number bounds

Lemma

(Upper bound) If $|E_D v|^2_{\widetilde{S}_\Gamma} \le C_E |v|^2_{\widetilde{S}_\Gamma} \quad \forall v \in \widetilde{W}_\Gamma$, *then*

$$u^T \widetilde{S}_\Gamma u \le C_E u^T M_{BDDC} u \quad \forall u \in \widetilde{W}_\Gamma. \tag{2}$$

Proof.

$$u^T \widetilde{S}_\Gamma u = u^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma u = u^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma M_{BDDC}^{-1} M_{BDDC} u$$

$$
\begin{aligned}
&= u^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w \\
&\le (\widetilde{R}_\Gamma u, \widetilde{R}_\Gamma u)^{1/2}_{\widetilde{S}_\Gamma} (E_D \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w, E_D \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w)^{1/2}_{\widetilde{S}_\Gamma} \\
&\le (u^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma u)^{1/2} C_E^{1/2} (\widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w, \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w)^{1/2}_{\widetilde{S}_\Gamma} \\
&= C_E^{1/2} (u^T \widetilde{S}_\Gamma u)^{1/2} (u^T M_{BDDC} u)^{1/2},
\end{aligned}
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- Also work on DG by Dryja and Sarkis and by Chung and Kim.

Stiffness scaling in frequency space

- New average operator E_D across a face $F \subset \Gamma$, common to two subdomains Ω_i and Ω_j, defined in terms of two Schur complements:

$$S_{FF}^{(k)} := A_{FF}^{(k)} - A_{FI}^{(k)} {A_{II}^{(k)}}^{-1} A_{IF}^{(k)}, \;\; k = i, j.$$

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- The deluxe averaging operator is then defined by

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- Similar formulas for subdomain edges and other equivalence classes of interface variables. The operator E_D is assembled from these components.

BDDC deluxe

- As we have shown, the core of any estimate for a BDDC algorithm is the norm of the average operator E_D. Known for FETI–DP since 2002,

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- Analysis of traditional BDDC requires an extension theorem; deluxe version does not.
- In addition to this good choice of averaging, a good primal space has to be chosen.

Analysis of BDDC deluxe

- Instead of estimating $(R_F^T \bar{w}_F)^T S^{(i)} R_F^T \bar{w}_F$, estimate the norm of $R_F^T(w^{(i)} - \bar{w}_F)$. By simple algebra, we find that

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- By more algebra, noting that $R_F S^{(i)} R_F^T = S_{FF}^{(i)}$:

$$(R_F^T (w_F^{(i)} - \bar{w}_F))^T S^{(i)} (R_F^T (w_F^{(i)} - \bar{w}_F)) =$$

$$(w_F^{(i)} - w_F^{(j)})^T S_{FF}^{(j)} (S_{FF}^{(i)} + S_{FF}^{(j)})^{-1} S_{FF}^{(i)} (S_{FF}^{(i)} + S_{FF}^{(j)})^{-1} S_{FF}^{(j)} (w_F^{(i)} - w_F^{(ji)})).$$

  Add similar expression for the subdomain Ω_j. More algebra gives the bound:

$$(w_F^{(i)} - w_F^{(j)})^T S_{FF}^{(i)} : S_{FF}^{(j)} (w_F^{(i)} - w_F^{(ji)}))$$

  where

$$A : B := (A^{-1} + B^{-1})^{-1}$$

  is known as a parallel sum.

Proof continued

Lemma

Let A and B be two symmetric positive semi-definite matrices of the same order. Then, $z^T A : Bz = \inf_{z=x+y}(x^T Ax + y^T By)$.

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- Routine for H^1: completely standard estimates in the domain decomposition literature: Face lemma; compare energy of zero extension of face values by zero with that of the minimal energy extension. Factor of $C(1 + \log(H/h))^2$ results.
- Also estimate contributions from subdomain edges, etc. For many edges, more than two Schur complements enters in in the averages. Bound but with small integer factors.

Adaptive choice of primal spaces

- The subdomain bounds can be expressed in terms of Schur complements of Schur complements

$$\breve{S}_{FF}^{(i)} := S_{FF}^{(i)} - S_{F'F}^{(i)T} S_{F'F'}^{(i)-1} S_{F'F}^{(i)}$$

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- Similar devices, so far not equally appealing, have been developed for equivalence classes with more than two elements. Pechstein and Dohrmann ETNA 46 pp. 273-336.
- For $H(\text{div})$ there are no such classes; Oh et al, Math. Comp. 87(310). Works well for nasty coefficients.

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- This problem has been and remains a focus of many studies of BDDC and FETI–DP.
- One way of dealing with this is to introduce a third or even more levels. Very interesting proof that it works well is due to Xuemin Tu. PETSc code by Stefano Zampini has been quite successful and has been combined with adaptive choices of primal spaces, e.g., for $H(\text{div})$ and $H(\mathbf{curl})$ problems.

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- Choosing a standard primal space based on vertex variables only leads to poor performance for 3D. However, in a paper, in progress by Dohrmann, Pierson, and OBW, it will be shown that in many cases, the primal problem can be solved approximately and very effectively using a coarse space of that same small dimension.

Outline of recent work

- A two-level BDDC preconditioner in additive form:

$$M^{-1} = M_{local}^{-1} + \Phi_D K_c^{-1} \Phi_D^T,$$

where K_c is the coarse matrix and Φ_D is a weighted interpolation matrix.

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- Let M_c^{-1} be a preconditioner for K_c and for $0 < \beta_1 \leq \beta_2$

$$\beta_1 u_c^T K_c^{-1} u_c \leq u_c^T M_c^{-1} u_c \leq \beta_2 u_c^T K_c^{-1} u_c \quad \forall u_c.$$

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- Choose $M_c^{-1} = \Psi K_{cr}^{-1} \Psi^T + \text{diag}(K_c)^{-1}$ where $K_{cr} := \Psi^T K_c \Psi$. (Point Jacobi + coarse correction.)

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- The hard part is to estimate β_1; for an important case, the result by Tu and ultimately Brenner again comes into play.

Almost incompressible elasticity

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- If the pressure is discontinuous, the elasticity problem can be reduced to a positive definite problem by eliminating the pressure on the element level.
- Discrete saddle point system:

$$\begin{bmatrix} \mu A & B^T \\ B & -\frac{1}{\lambda} C \end{bmatrix} \begin{bmatrix} \mathbf{u}_h \\ p_h \end{bmatrix} = \begin{bmatrix} f_h \\ 0 \end{bmatrix},$$

where A, B, and C are the matrices associated with the bilinear forms of the mixed formulation of almost incompressible elasticity. μ and λ the Lamé parameters.

Block FETI-DP preconditioner a la Tu and Li, 2015

- Split displacement into interior, dual, and primal components:

$$\mathbf{u}_h = (\mathbf{u}_I, \mathbf{u}_\Delta, \mathbf{u}_\Pi),$$

and pressure into interiors and, interface components:

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and introduce $\lambda_\Delta =$ Lagrange multipliers used to enforce the continuity of the dual displacements in the limit.

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- Permute the saddle point system to the form:

$$\begin{bmatrix} \mu A_{II} & B_{II}^T & \mu A_{I\Delta} & \mu A_{I\Pi} & B_{\Gamma I}^T & 0 \\ B_{II} & -\frac{1}{\lambda} C_{II} & B_{\Delta I}^T & B_{\Pi I}^T & -\frac{1}{\lambda} C_{I\Gamma} & 0 \\ \mu A_{\Delta I} & B_{\Delta I} & \mu A_{\Delta\Delta} & \mu A_{\Delta\Pi} & B_{\Gamma\Delta}^T & B_\Delta^T \\ \mu A_{\Pi I} & B_{\Pi I} & \mu A_{\Pi\Delta} & \mu A_{\Pi\Pi} & B_{\Gamma\Pi}^T & 0 \\ B_{\Gamma I} & -\frac{1}{\lambda} C_{\Gamma I} & B_{\Gamma\Delta} & B_{\Gamma\Pi} & -\frac{1}{\lambda} C_{\Gamma\Gamma} & 0 \\ 0 & 0 & B_\Delta & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \\ \mathbf{u}_\Pi \\ p_\Gamma \\ \lambda_\Delta \end{bmatrix} = \begin{bmatrix} f_I \\ 0 \\ f_\Delta \\ f_\Pi \\ 0 \\ 0 \end{bmatrix}$$

Block FETI-DP preconditioners

- After a block Gaussian elimination step, a Schur complement $-G$, which is negative definite and a reduced system.

FETI-DP type system

$$G \begin{bmatrix} p_\Gamma \\ \lambda_\Delta \end{bmatrix} = g. \tag{3}$$

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$M_{p_\Gamma}^{-1} = \mu$ times the inverse of Schur complement of C or use BDDC. $M_{\lambda_\Delta}^{-1}$ essentially the same as for compressible elasticity.