The Riemann Hypothesis and computers

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November 1, 2018

Riemann, 1850s:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}, \ \mathrm{Re}(s) > 1\,.$$

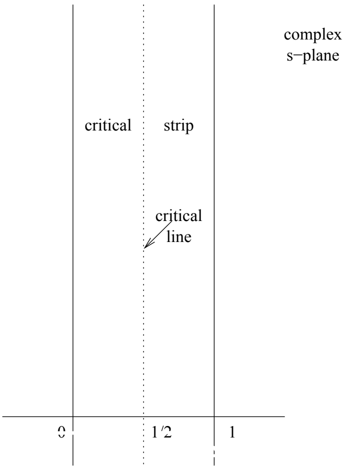
Showed $\zeta(s)$ can be continued analytically to $\mathbb{C} \setminus \{1\}$ and has a first order pole at $s = 1$ with residue 1. If

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)\ ,$$

then (functional equation)

$$\xi(s) = \xi(1-s)\ .$$

Critical strip:



Riemann, 1859:

- almost all nontrivial zeros of the zeta function are on the critical line (positive assertion, no hint of proof)
- it is likely that all such zeros are on the critical line (now called the Riemann Hypothesis, RH)
- (ambiguous: cites computations of Gauss and others, not clear how strongly he believed in it) $\pi(x) < \mathrm{Li}(x)$

Rigorous zero determination:

- $\xi(s)$ real on critical line
- sign changes of $\xi(s)$ come from zeros on the line
- principle of the argument gives total number of zeros, so if zeros simple and on the line, can establish that rigorously (subject to correctness of algorithms, software, and hardware)

Numerical verifications of RH for first n zeros:

Riemann 1859	?
Gram 1903	15
...	
Hutchinson 1925	138
Titchmarsh et al. 1935/6	1,041
Turing 1950 (published 1953)	1,054
...	
te Riele et al. 1986	1,500,000,000
...	
Gourdon 2004	10,000,000,000,000

Large blocks of zeros at large heights:

O.	10^{23}
Gourdon	10^{24}
Hiary	10^{28}

Bober & Hiary: small blocks at 10^{36}

10^{23} denotes zero number 10^{23}, not height

Algorithms for verifying RH for first n zeros:

Euler–Maclaurin	$n^{2+o(1)}$
Riemann–Siegel	$n^{\frac{3}{2}+o(1)}$
O.–Schönhage	$n^{1+o(1)}$

Algorithms for single values of zeta at height t:

Euler–Maclaurin	$t^{1+o(1)}$
Riemann–Siegel	$t^{1/2+o(1)}$
Schönhage	$t^{3/8+o(1)}$
Heath-Brown	$t^{1/3+o(1)}$
Hiary	$t^{4/13+o(1)}$

Interest in zeros of zeta function:

- numerical verification of RH
- $\pi(x) - \mathrm{Li}(x)$ and related functions
- (more recently) distribution questions related to hypothetical random matrix connections

Riemann and Ingham:

Riemann:

$$\pi(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) + O(x^{1/2}\log x)$$

Ingham:

certain averages of $(\pi(x) - \mathrm{Li}(x))$ = nice sums over ρ

Riemann and Ingham:

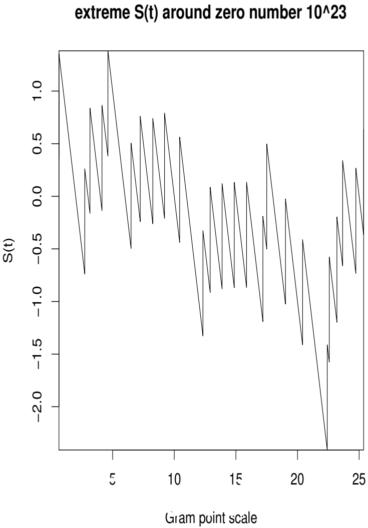
- Littlewood (1914): Riemann "conjecture" that $\pi(x) < \mathrm{Li}(x)$ false
- Skewes (1933, assuming RH): first counterexample $< 10^{10^{10^{34}}}$
- Ingham approach (with extensive computations but without producing explicit counterexample): $< 10^{317}$

Beware the law of small numbers (especially in number theory):

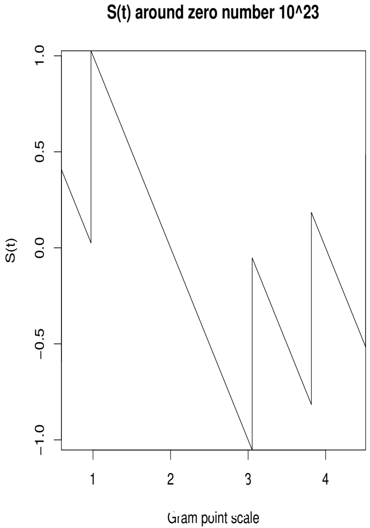
$N(t) = 1 + \frac{1}{\pi}\theta(t) + S(t)$
where $\theta(t)$ is a smooth function, and $S(t)$ is small:

- $|S(t)| = O(\log t)$
- $$\frac{1}{t}\int_{10}^{t} S(u)^2 du \sim c \log\log t$$
- $$|S(t)| < 1 \quad \text{for} \quad t < 280$$
- $$|S(t)| < 2 \quad \text{for} \quad t < 6.8 \times 10^6$$
- largest observed value of $|S(t)|$ just 3.3455

extreme among 10^6 zeros near zero 10^{23}:



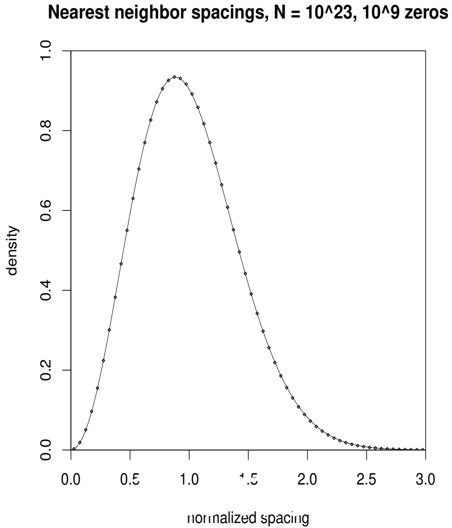
Typical behavior of S(t):



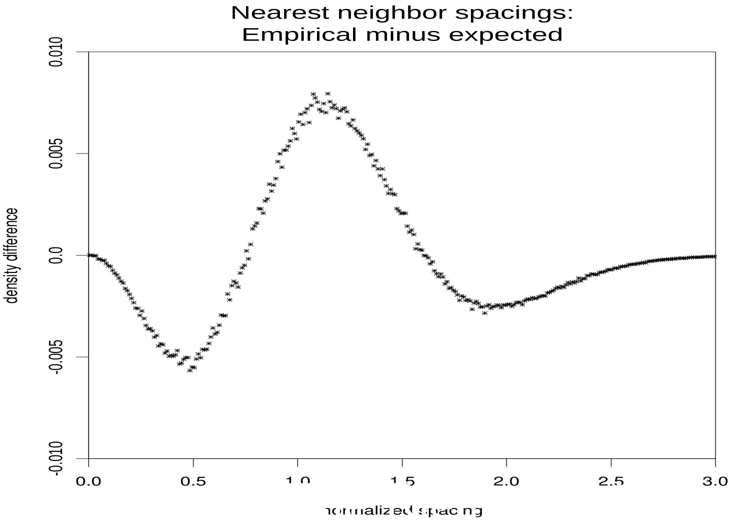
Where should one look for counterexamples to RH?

- could argue that need to reach regions where $S(t)$ is routinely over 100
- requires $t \sim 10^{10^{10,000}}$
- cannot even specify such heights with available or conceivable technology
- and counterexamples are likely rare!

Zeta zeros and random matrices:



Zeta zeros and random matrices:



Conclusion (from a sign in a computing support office):

- We are sorry that we have not been able to solve all of your problems, and we realize that you are about as confused now as when you came to us for help. However, we hope that you are now confused on a higher level of understanding than before.