Circle patterns on surfaces with complex projective structures

Joint work with Andrew Yarmola

Jean-Marc Schlenker

University of Luxembourg

Circle Packings and Geometric Rigidity
ICERM, July 6-10, 2020 (online)

Where do circles live ?

What do we need to consider circles ?

- The Euclidean plane.

Circles are invariant under isometries $\Rightarrow$ also in Euclidean surfaces.
Flat surface : charts in $\mathbb{R}^2$, transitions maps are Euclidean isometries.

- The hyperbolic plane.

Same reason – also on hyperbolic surfaces.
Hyperbolic surface : charts in $\mathbb{H}^2$, transitions maps are hyperbolic isometries.

- $\mathbb{CP}^1$.

Notion of circle, invariant under Möbius transformations.
Complex projective structures : charts in $\mathbb{CP}^1$, transition maps in $PSL(2, \mathbb{C})$.
Also called $\mathbb{CP}^1$-structures on a surface S. Space $\mathcal{CP}_S$.

Complex projective structures on surfaces

Let $\sigma \in \mathcal{CP}_S$ be a $\mathbb{CP}^1$-structure on S. We have :

- A developing map $dev : \tilde{S} \to \mathbb{CP}^1$.
- A holonomy representation $\rho : \pi_1 S \to PSL(2, \mathbb{C})$.

σ is *Fuchsian* if *dev* is a homeomorphism onto a disk, or equivalently if ρ is Fuchsian (into $PSL(2, \mathbb{R})$, up to conjugation).
Examples :

- A hyperbolic structure determines a Fuchsian $\mathbb{CP}^1$-structure on S.
- An Euclidean structure on T^2 determines a $\mathbb{CP}^1$-structure, $dev(\tilde{T}^2) = \mathbb{CP}^1 \setminus \{\infty\}$.

Thm. (Thurston–Lok) $\mathbb{CP}^1$-structures are locally determined by their ~~developing map~~ $\rho : \pi_1 S \to PSL(2, \mathbb{C})$.

Therefore, $\mathcal{CP}_S$ has *complex* dimension $6g - 6$ for $g \geq 2$, 2 for $g = 1$.

Circle packings on surfaces with $\mathbb{CP}^1$-structures

S^2 admits a unique $\mathbb{CP}^1$-structure, given by $\mathbb{CP}^1$.

Thm. (Koebe) The 1-skeleton of a triangulation of S^2 is the incidence graph of a circle packing of $\mathbb{CP}^1$, unique up to Möbius transformations.

Thm. (Thurston) The 1-skeleton of a triangulation of S_g, $g \geq 2$, is the incidence graph of a unique circle packing in S_g equipped with *some hyperbolic* metric.

Question. How to understand all circle packings on S_g equipped with *any* $\mathbb{CP}^1$-structure, not necessarily Fuchsian ?

There should be many – real dimension $6g - 6$.

The KMT conjecture

Since $PSL(2,\mathbb{C})$ acts on $\mathbb{CP}^1$ by holomorphic maps, any $\mathbb{CP}^1$-structure on S determines an underlying *complex structure*.
Complex structure : charts in $\mathbb{C}$, transition maps holomorphic.
The space of complex structures on S (up to isotopy) is the *Teichmüller space* of S, $\mathcal{T}_S$. It has real dimension $6g-6$.
$\mathcal{CP}_S \simeq T^*\mathcal{T}_S$, through a construction using the Schwarzian derivative.

Kojima, Mizushima and Tan proposed :

Conj. (KMT) Let Γ be the 1-skeleton of a triangulation of S_g, let $\mathcal{CP}_\Gamma$ be the space of $\mathbb{CP}^1$-structures on S admitting a circle packing with incidence graph Γ. Then the forgetful map $\mathbb{CP}_\Gamma \to \mathcal{T}_S$ is a homeomorphism.

Holds for $g=0$ (Koebe), also for tori when Γ has only one vertex (KMT).
Note : interaction between discrete and continuous conformal structures.

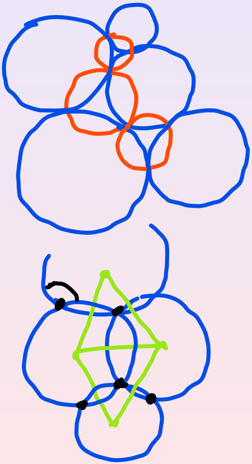
Delaunay circle patterns

A *Delaunay circle pattern* on S equipped with a $\mathbb{CP}^1$-structure S is (basically) the pattern of circles associated to the Delaunay decomposition of a finite set of points on S.

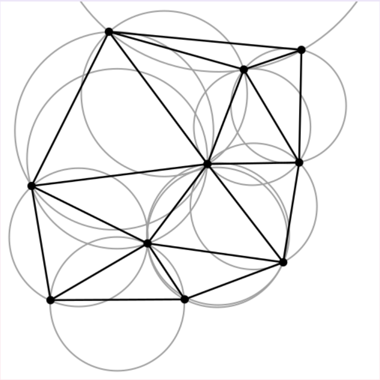
To a circle packing on (S, σ) with incidence graph the 1-skeleton of a triangulation, one can associate a Delaunay circle pattern with all intersection angles $\pi/2$: add *dual* circles, associated to the faces of Γ and orthogonal to the circles associated to adjacent vertices.

To a Delaunay circle pattern one can associate :

- An incidence graph (vertices=circles, edges=incidence relations),
- an angle for each edge : the intersection angle between circles (π if tangent).



A Delaunay circle pattern



The KMT conjecture for Delaunay circle patterns

The intersection angles of a Delaunay circle pattern satisfy :

1. For each vertex v of Γ, $\sum_{v\in e}\theta_e = 2\pi$.
2. For each closed contractible path in Γ^* not bounding a face, $\sum_e \theta_e > 2\pi$.

Conj A. Let Γ be the 1-skeleton of a cell decomposition of S, and $\theta : \Gamma^1 \to (0,\pi)$ satisfying (1) and (2). Let $\mathcal{CP}_{\Gamma,\theta}$ be the space of $\mathbb{CP}^1$-structures with a Delaunay circle pattern with incidence graph Γ and intersection angles θ. The forgetful map $\mathcal{CP}_{\Gamma,\theta} \to \mathcal{T}_S$ is a homeomorphism.

A deformation argument

A possible path towards a proof of Conj. A :

1. $\mathcal{CP}_{\Gamma,\theta}$ has real dimension $6g-6$,
2. $\pi_{|\mathcal{CP}_{\Gamma,\theta}}$ has injective differential (*infinitesimal rigidity*),
3. $\pi_{|\mathcal{CP}_{\Gamma,\theta}} : \mathcal{CP}_{\Gamma,\theta} \to \mathcal{T}_S$ is *proper*,
4. $\mathcal{CP}_{\Gamma,\theta}$ is connected and $\mathcal{T}_S$ simply connected.

(1)+(2) $\to$ $\pi_{|\mathcal{CP}_{\Gamma,\theta}}$ is a local homeomorphism,
(3) $\to$ it is a covering map,
(4) $\to$ the degree is 1.

For (2) see talk by Wayne Lam, for $g=1$.

Thm B. (3) holds.

Note. Also implies the corresponding properness for circle *packings* follows.

From $\mathbb{CP}^1$-structure to hyperbolic ends

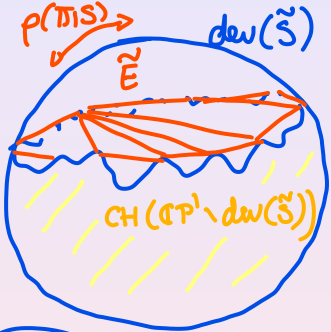
Def. A hyperbolic end is a hyperbolic manifold homeomorphic to $S \times [0, \infty)$, complete on the side of ∞, and bounded on the side of 0 by a concave pleated surface.

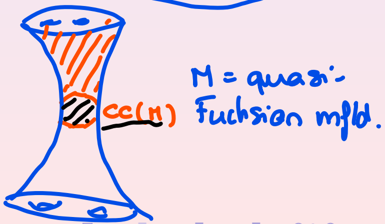
Thm. (Thurston) 1–1 correspondence between hyperbolic ends and $\mathbb{CP}^1$-structures on S.

Hyperbolic ends are also determined by the data on the 0 side : a hyperbolic metric and a *measured bending lamination*.

$\mathcal{CP}_S \simeq \mathcal{T}_S \times \mathcal{ML}_S$.

Delaunay circle pattern at infinity $\rightarrow$ ideal polyhedron in E, ext. dihedral angles θ.





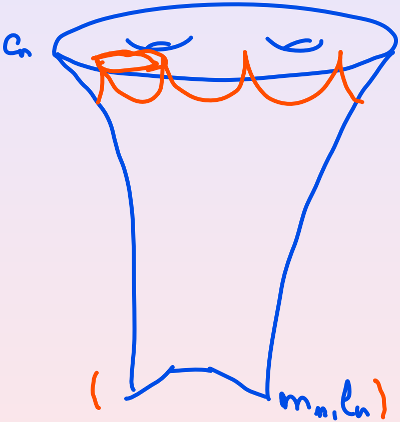
Key ideas of the proof of Thm B

Let $\sigma_n \in \mathcal{CP}_{\Gamma,\theta}, n \in \mathbb{N}$, and let $c_n = \pi(\sigma_n)$. We assume that $(c_n)_{n\in\mathbb{N}}$ converges, and need to prove that a subsequence of $(\sigma_n)_{n\in\mathbb{N}}$ converges.

We consider the hyperbolic end E_n associated to σ_n, and $(m_n, l_n) \in \mathcal{T}_S \times \mathcal{ML}_S$.

Then l_n is bounded because dihedral angles are bounded,

m_l is bounded because c_n is bounded.



The Weyl problem in $\mathbb{H}^3$ and its dual

Weyl problem. (Alexandrov, Pogorelov) Let g be a metric on S^2 with $K \geq -1$. Is there a unique convex body in $\mathbb{H}^3$ with induced metric g on its boundary ?

Weyl* problem. Let g be a metric on S^2 with $K < 1$ and closed geodesics of length $L > 2\pi$. Is there a unique convex body in $\mathbb{H}^3$ with $III = g$ on the boundary ?

N on $S \subset \mathbb{H}^3$

$III(x,y) = \langle \nabla_x N, \nabla_y N \rangle$

For polyhedra, *III* is related to dihedral angles.
Results on *Weyl** for compact polyhedra (Rivin-Hodgson), ideal polyhedra (Rivin), smooth surfaces (S.) etc.
For Fuchsian polyhedra (Bobenko-Springborn, Fillastre, Leibon, ...)

The Weyl problem in hyperbolic ends

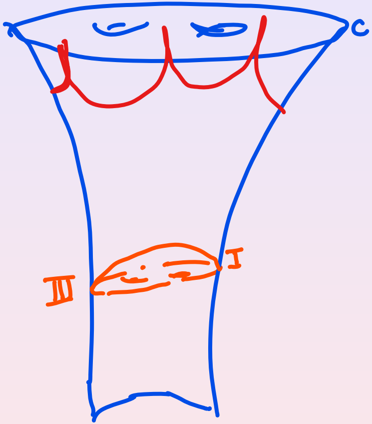
Question. Let g be a metric on S with $K \geq -1$, and let $c \in \mathcal{T}_S$. Is there a unique hyperbolic end containing a convex domain with induced metric g on the boundary, and with conformal structure at infinity c ?

Question*. Let g be a metric on S with

$K < 1$ and closed, contractible geodesics of length $L > 2\pi$, and let $c \in \mathcal{T}_S$. Is there a unique hyperbolic end containing a convex domain with $III = g$ on the boundary, and with conformal structure at infinity c ?

Conj. A is a special case of the second

question for "ideal polyhedra".



Unbounded convex subsets in $\mathbb{H}^3$

Consider $\tilde{E}$, and forget the group action. Leads to a Weyl problem for unbounded convex domains in $\mathbb{H}^3$. Different flavors, one particularly connected to Conj. A.

Question. Let g be a complete metric of $K \in (-1, 0)$ on D^2, and let $u : \partial_\infty(D^2, g) \rightarrow \partial D^2$ be quasi-symmetric. Is there a unique properly immersed convex disk in $\mathbb{H}^3$ with induced metric g and with u as the gluing map with the boundary at infinity facing it ?

Question*. Let g be a complete metric of $K < 1$ on D^2, with closed geodesics of $L > 2\pi$, and let $u : \partial_\infty(D^2, g) \rightarrow \partial D^2$ be quasi-symmetric. Is there a unique properly immersed convex disk in $\mathbb{H}^3$ with $III = g$ and with u as the gluing map with the boundary at infinity facing it ?