The spectrum of the Laplacian in a domain bounded by a flexible polyhedron in Euclidean space does not always remain unaltered during the flex

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Abstract & References

We study the Dirichlet and Neumann eigenvalues for the Laplace operator in bounded domains of Euclidean d-space whose boundary is a flexible polyhedron. The main result is that both the Dirichlet and Neumann spectra of the Laplace operator in such a domain do not necessarily remain unaltered during the flex of its boundary.

The talk is based on the article: V. Alexandrov. *The spectrum of the Laplacian in a domain bounded by a flexible polyhedron in* $\mathbb{R}^d$ *does not always remain unaltered during the flex.* Journal of Geometry, **111**, no. 2. Paper No. 32 (2020).

What is a polyhedron

In this talk, a *polyhedron* is a connected boundary-free compact polyhedral $(d-1)$-manifold in $\mathbb{R}^d$, $d \geqslant 2$. Self-intersections of any type are not excluded.

If the boundary of a bounded connected open set $D \subset \mathbb{R}^d$ is a polyhedron P, we write $D = [\![P]\!]$ and say that D is the domain bounded by the polyhedron P.

What is a flexible polyhedron

A polyhedron P_0 is called *flexible* if its spatial shape can be changed continuously by changing its dihedral angles only.

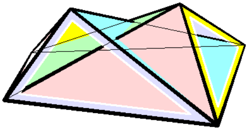
In other words, P_0 is flexible if
• P_0 belongs to a continuous family $\{P_t\}_{t\in[0,1]}$ of polyhedra P_t, such that each face of P_t is congruent to the corresponding (by continuity) face of P_0; and
• P_0 and P_t are not congruent to each other for all $0 < t \leqslant 1$.

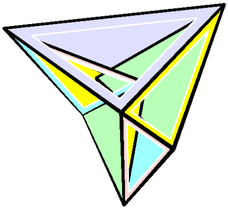
The above-mentioned continuous family $\{P_t\}_{t\in[0,1]}$ is called the *flex* of the polyhedron P_0.

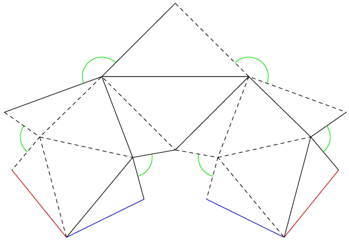
A polyhedron is called *rigid* if it is not flexible.

Some basic facts about flexible polyhedra: slide 1

(a) flexible polyhedra *do exist* (R. Bricard, 1897) and (R. Connelly, 1977); moreover, they can have any genus and can be non-orientable (M.I. Shtogrin, 2015);







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Some basic facts about flexible polyhedra: slide 2

(b) flexible polyhedra are rare objects:

- every compact *convex* polyhedron is rigid (A.L. Cauchy, 1813); and
- *almost all* simply connected polyhedra in $\mathbb{R}^3$ are rigid (H. Gluck, 1975);

(c) for every flex, every orientable flexible polyhedron necessarily keeps unaltered the *total mean curvature* (R. Alexander, 1985); i. e., the quantity

$$\sum_{\ell} |\ell|(\pi - \alpha(\ell))$$

remains constant during every flex, where $|\ell|$ is the length of the edge ℓ, $\alpha(\ell)$ is the value of the interior dihedral angle at the edge ℓ, and the sum extends to all edges of the polyhedron;

Some basic facts about flexible polyhedra: slide 3

(d) for every flex, every orientable flexible polyhedron necessarily keeps unaltered the *volume* of the domain they bound (for $\mathbb{R}^3$: I.Kh. Sabitov, 1996 and R. Connelly et al., 1997; for $\mathbb{R}^d$, $d \geqslant 4$: A.A. Gaifullin, 2014);

(e) for every flex, every orientable flexible polyhedron necessarily keeps unaltered the *Dehn invariants* (A.A. Gaifullin & L.S. Ignashchenko, 2018); i. e., the quantity

$$\sum_{\ell} |\ell| f(\varphi(\ell))$$

remains constant during every flex, where $|\ell|$ is the length of the edge ℓ, $\varphi(\ell)$ is the value of the interior dihedral angle at the edge ℓ, $f : \mathbb{R} \to \mathbb{R}$ is a $\mathbb{Q}$-linear fuction suth that $f(\pi) = 0$, and the sum extends to all edges of the polyhedron;

Some basic facts about flexible polyhedra: slide 4

(f) flexible polyhedra do exist in all *spaces of constant curvature* of dimension $\geqslant 3$ and in *pseudo-Euclidean spaces* of dimension $\geqslant 3$; moreover, in many of these spaces they possess properties similar to properties (a)–(e).

The problem we are studying

Being motivated by the properties (c), (d), and (e), we would like to find new invariants of flexible polyhedra in $\mathbb{R}^d$, $d \geqslant 3$, that is, quantities which are preserved under every flex.

In our opinion, it is natural to check for the role of such invariants the Dirichlet and Neumann eigenvalues of the Laplace operator in the domain $[\![P_0]\!] \subset \mathbb{R}^d$, bounded by the flexible polyhedron P_0, because:

• the statement that the spectrum of the Laplacian remains unaltered during the flex agrees with the Weyl law on the asymptotics of eigenvalues of the Laplacian;

• if the spectrum of the Laplacian remains unaltered during the flex, the Weyl law provides us with a new proof of the Bellows Conjecture.

Recall the Weyl law

The *Weyl law* reads that, under certain assumptions on the boundary $\partial\Omega$ of a bounded domain $\Omega \subset \mathbb{R}^d$, the following asymptotic formula holds true for $k \to \infty$:

$$\mathcal{N}(k) = \frac{\mathrm{vol}_d(\Omega)}{\Gamma\left(\frac{d+2}{2}\right)}\left(\frac{k}{2\sqrt{\pi}}\right)^d \mp \frac{\mathrm{vol}_{d-1}(\partial\Omega)}{4\Gamma\left(\frac{d+1}{2}\right)}\left(\frac{k}{2\sqrt{\pi}}\right)^{d-1} + o(k^{d-1}).$$

Here $\mathcal{N}(k)$ is the *eigenvalue counting function*, that is the number of eigenvalues, which do not exceed k^2 (repeating each eigenvalue according to its multiplicity), vol_p denotes the *p-dimensional volume* of a set, and Γ denotes the *Euler gamma function*.
The minus sign corresponds to the Dirichlet problem ($\Delta u = -\nu^2 u$ in Ω, $u|_{\partial\Omega} = 0$), while the plus sign corresponds to the Neumann problem ($\Delta u = -\nu^2 u$ in Ω, $\frac{\partial u}{\partial \boldsymbol{n}}|_{\partial\Omega} = 0$).

The main result

Theorem (Alexandrov, 2020)

For every $d \geqslant 3$, $\varepsilon > 0$, and every embedded flexible polyhedron $P_0 \subset \mathbb{R}^d$ there is an embedded flexible polyhedron $\widetilde{P}_0 \subset \mathbb{R}^d$ and its flex $\{\widetilde{P}_s\}_{s\in[0,1)}$ such that

• *the combinatorial structure of $\widetilde{P}_0$ is a subdivision of the combinatorial structure of P_0;*

• *the Hausdorff distance between the sets $\widetilde{P}_0$ and P_0 is less than ε;*

• *both Dirichlet and Neumann spectra of the d-dimensional Laplacian in the domain $[\![\widetilde{P}_s]\!] \subset \mathbb{R}^d$ do not remain unaltered when s changes in the interval $[0, 1)$.*

The proof is based on the following vesrion of the Weyl law:

Theorem (Fedosov, Sov. Math., Dokl. **5**, 988–990 (1964)):

Let $d \geqslant 2$, $0 \leqslant p \leqslant d-1$, and let a bounded domain $D \subset \mathbb{R}^d$ be such that its boundary ∂D is a polyhedron. Let $\{F_i^{d-2}\}_i$ be the set of all $(d-2)$-dimensional faces of ∂D, and let φ_i stand for the value of the dihedral angle of D at F_i^{d-2}. Then the following asymptotic formula, involving the eigenvalue counting function $\mathcal{N}(k)$, holds true as $k \to \infty$ for both the Dirichlet and Neumann problems:

$$\frac{1}{\Gamma(p+1)} \int_0^k (k-\tau)^p \, d\mathcal{N}(\tau) = \sum_{l=1}^{d} a_l \frac{\Gamma(l+1)}{\Gamma(p+l+1)} k^{p+l} + O(k^{d-1}).$$

Theorem (Fedosov, Sov. Math., Dokl. **5**, 988–990 (1964)) – continuation from the previous slide:

The coefficients a_d, a_{d-1}, and a_{d-2} are given by the following explicit formulas:

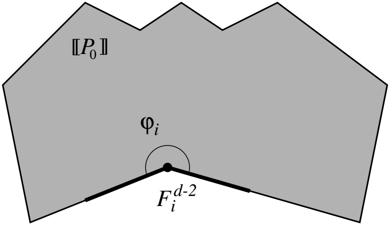
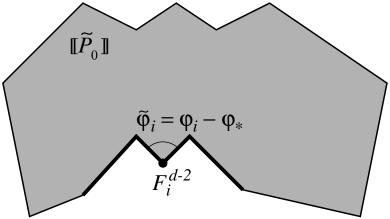
$$\begin{aligned} a_d &= \frac{\mathrm{vol}_d(D)}{2^d \pi^{d/2} \Gamma\left(\frac{d}{2}+1\right)}, \\ a_{d-1} &= \mp \frac{\mathrm{vol}_{d-1}(\partial D)}{2^{d+1} \pi^{(d-1)/2} \Gamma\left(\frac{d+1}{2}\right)}, \qquad (*) \\ a_{d-2} &= \frac{1}{2^{d+1} \pi^{d/2} \Gamma\left(\frac{d}{2}\right)} \sum_i \frac{\varphi_i^2 - \pi^2}{3\varphi_i} \mathrm{vol}_{d-2}\left(F_i^{d-2}\right). \end{aligned}$$

In the formula $(*)$, the minus sign corresponds to the Dirichlet problem, while the plus sign corresponds to the Neumann problem.

The above theorem was a part of Ph.D. thesis of Professor Boris V. Fedosov (1938–2011), a well-known Moscow mathematician, who made significant contribution to the theory of partial differential equations and differential geometry, including index theory and deformation quantization.

You can find more details about his life and scientific heritage in his obituary

M.S. Agranovich, L.A. Aĭzenberg, G.L. Alfimov, M.I. Vishik, et al.
Boris Vasil'evich Fedosov (obituary).
Russian Mathematical Surveys. **67**, 167–174 (2012).


⟦P_0⟧
φ_i
$F_i^{d\text{-}2}$

⟦$\widetilde{P}_0$⟧
$\widetilde{\varphi}_i = \varphi_i - \varphi_*$
$F_i^{d\text{-}2}$

Back to the main result

Theorem (Alexandrov, 2020)

For every $d \geqslant 3$, $\varepsilon > 0$, and every embedded flexible polyhedron $P_0 \subset \mathbb{R}^d$ there is an embedded flexible polyhedron $\widetilde{P}_0 \subset \mathbb{R}^d$ and its flex $\{\widetilde{P}_s\}_{s\in[0,1)}$ such that

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Conclusion

We knew before that the total mean curvature, volume, and Dehn invariants of every flexible polyhedron are preserved during its flexes.

Now we know that, for some flexible polyhedra, eigenvalues of the Laplace operator are nonconstant during flexes.

Everybody is welcome to look for a new nontrivial geometric quantity corresponding to a flexible polyhedron in $\mathbb{R}^3$, such that this quantity remains constant during all its flexes.

The end.

Thank you for attention!