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Joint work with Andy Kustin and Rebecca R.G.

Problem:

$\mathbf{k}$ = field of char. zero, $P = \mathbf{k}[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}]$, $N, n \geq 1$ integers

We study the minimal free resolution of R over P, *where*

$$I = (x^N, y^N, z^N, w^N) : (x^n + y^n + z^n + w^n)$$

$$R = P/I$$

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This can then be used to build the $P/(x^n + y^n + z^n + w^n)$-resolution of

$$P/(x^N, y^N, z^N, w^N, x^n + y^n + z^n + w^n)$$

Notation

Let $N = dn + r$*, with* $0 \leq r \leq n - 1$*.*
The answer will be given in terms of d *and* r *instead of* n *and* N*.*

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$A^t : F_4 = P(-s-4) \to F_3$ *and* $A : F_1 \to P$ *preserve degrees; it follows that*

$$F_3 = P(-s-4+d_1) \oplus \cdots P(-s-4+d_k)$$

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To find the graded free module F_2**:** in addition to the above, we also need to know the Hilbert function of $R = P/I$.

Definition

$$H(n) := \dim_k P_n = \binom{n+3}{3}$$

For any graded P-module $M = \bigoplus_n M_n$,

$$H_M(n) := \dim_k(M_n)$$

Once the Hilbert function H_R and the free graded modules F_1, F_3, F_4 are known, we have

$$H_{F_2} = H_R - H + H_{F_1} + H_{F_3} - H_{F_4}$$

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Proof: Let

$$F_2 = P(-\delta_1)^{b_1} \oplus \cdots \oplus P(-\delta_l)^{b_l}$$

with $\delta_1 < \delta_2 < \cdots < \delta_l$, and $b_1, \ldots, b_l \geq 1$.

Plugging in values of n in the Hilbert function, we have

$$H_{F_2}(n) = b_1 H(n - \delta_1) + \cdots + b_l H(n - \delta_l) \quad \text{for all } n \geq 0$$

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$H(n-\delta_i) = \begin{cases} 0 & \text{for} \quad n < \delta_i \\ 1 & \text{for} \quad n = \delta_i \end{cases}$, so we obtain:

$$\delta_1 = \min\{n \,|\, H_{F_2}(n) \neq 0\}, \qquad b_1 = H_{F_2}(\delta_1)$$

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$$\delta_1 = \min\{n \,|\, H_{F_2}(n) \neq 0\}, \qquad b_1 = H_{F_2}(\delta_1)$$

$$\delta_2 = \min\{n \,|\, H_{F_2}(n) > b_1 H(n-\delta_1)\},$$
$$b_2 = H_{F_2}(\delta_2) - b_1 H(\delta_2 - \delta_1)$$

ETC.

Summary

In order to find the graded Betti numbers in the resolution of $R = P/I$ over P, it suffices to know:

- *The socle degree of I*
- *The degrees of the generators of I*
- *The Hilbert function of R*

Observation

The socle degree of I is $s = 4N - 4 - n$.

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Proof: Recall $I = (x^N, y^N, z^N, w^N) : (x^n + y^n + z^n + w^n)$.
There is an injective homorphism:

$$R = \frac{P}{I} \hookrightarrow \frac{P}{(x^N, y^N, z^N, w^N})$$

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This raises degrees by n, and sends the socle of I to the socle of (x^N, y^N, z^N, w^N), which is $(xyzw)^{N-1}$.

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Definition

$\mathbf{G} = \mathbf{Z} \times \mathbf{Z_n} \times \mathbf{Z_n} \times \mathbf{Z_n} \times \mathbf{Z_n}$

Let $D \in \mathbf{Z}$ and $(\bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4) \in \mathbf{Z}_n \times \mathbf{Z}_n \times \mathbf{Z}_n \times \mathbf{Z}_n$,

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$P_{(D,\bar{r}_1,\bar{r}_2,\bar{r}_3,\bar{r}_4)} =$ the $\mathbf{k}$-span of the monomials $x^{\rho_1} y^{\rho_2} z^{\rho_3} w^{\rho_4}$ such that:

- $\rho_1 + \rho_2 + \rho_3 + \rho_4 = D$, and
- the image of ρ_i in $\mathbf{Z}_n$ is $\bar{r}_i$ for each $i = 1, \ldots 4$.

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- the image of ρ_i in $\mathbf{Z}_n$ is $\bar{r}_i$ for each $i = 1, \ldots 4$.

Note: $P_{(D,\bar{r}_1,\bar{r}_2,\bar{r}_3,\bar{r}_4)} = 0$ *unless* $\bar{D} = \bar{r}_1 + \bar{r}_2 + \bar{r}_3 + \bar{r}_4$.

Observation

P is graded by $\mathbf{G}$ in the sense that

$$P_{m_1} \cdot P_{m_2} \subseteq P_{m_1+m_2} \quad \text{for all } m_1, m_2 \in \mathbf{G}.$$

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$$\deg(x^N) = (N, \overline{r}, 0, 0, 0) \quad \deg(y^N) = (N, 0, \overline{r}, 0, 0)$$

$$\deg(z^N) = (N, 0, 0, \overline{r}, 0), \quad \deg(w^N) = (N, 0, 0, 0, \overline{r})$$

$$\deg(x^n + y^n + z^n + w^n) = (n, 0, 0, 0, 0)$$

- $I = (x^N, y^N, z^N, w^N) : (x^n + y^n + z^n + w^n)$ *is homogeneous under the multi-grading by* $\mathbf{G}$.

- *the multi-grading is inherited by* $R = P/I$.

Definition

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for each $g \in \mathbf{G}$, $H_M(g)$ is the vector space dimension of the component of M of degree g.

We find

- *the multi-degree of the socle of* I
- *the multi-degrees of the generators of* I
- *the multi-graded Hilbert function of* R

This information allows us to find multi-graded Betti numbers of the P*-resolution of* R*.*

Recall the multiplication by $x^n + y^n + z^n + w^n$

$$R = \frac{P}{I} \hookrightarrow \frac{P}{(x^N, y^N, z^N, w^N)}$$

sends the socle of I to $x^{N-1}y^{N-1}z^{N-1}w^{N-1}$, which has multi-degree
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Therefore, the socle of I has multi-degree
$(4N-4-n, \overline{r-1}, \overline{r-1}, \overline{r-1}, \overline{r-1})$

Notation

For a $g \in P$, $g^{[n]} \in P$ *is obtained by replacing* x, y, z, w *in* g *by* x^n, y^n, z^n, w^n.

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Let $D = nk + r_1 + r_2 + r_3 + r_4$ *with* $k \in \mathbf{Z}$,
$m = (D, \bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4) \in \mathbf{Z} \times \mathbf{Z}_n \times \mathbf{Z}_n \times \mathbf{Z}_n \times \mathbf{Z}_n$.

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The elements of P_m *have the form*

$$x^{r_1} y^{r_2} z^{r_3} w^{r_4} g^{[n]} \qquad \text{with } g \in P \text{ of degree } k$$

Finding generators - First simplification

Finding generators for I reduces to finding generators for the ideals

$$(x^{d+\epsilon_1}, y^{d+\epsilon_2}, z^{d+\epsilon_3}, w^{d+\epsilon_4}) : (x+y+z+w)$$

for each choice of $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \{0,1\}^4$.

Lemma

The elements of $I = (x^N, y^N, z^N, w^N) : (x^n + y^n + z^n + w^n)$ *of degree* $m = (D, \bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4)$ *have the form*

$$x^{r_1} y^{r_2} z^{r_3} w^{r_4} g^{[n]}$$

with $g \in (x^{d+\epsilon_1}, y^{d+\epsilon_2}, z^{d+\epsilon_3}, w^{d+\epsilon_4}) : (x + y + z + w)$*, where*

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with $g \in (x^{d+\epsilon_1}, y^{d+\epsilon_2}, z^{d+\epsilon_3}, w^{d+\epsilon_4}) : (x + y + z + w)$*, where*

$$\epsilon_i = \begin{cases} 1, & \text{if } r_i < r, \\ 0, & \text{otherwise,} \end{cases}$$

Corollary

The ideal I is generated by is generated by all the elements of the form

$$(x^{1-\epsilon_1}y^{1-\epsilon_2}z^{1-\epsilon_3}w^{1-\epsilon_4})^r g^{[n]}$$

with $g \in (x^{d+\epsilon_1}, y^{d+\epsilon_2}, z^{d+\epsilon_3}, w^{d+\epsilon_4}) : f$*,*
for all choices of $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \{0,1\}^4$*.*

Observation

The list of generators of I obtained above is redundant.

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A minimal set of generators consists of the generators obtained as above for $\sum_{i=1}^{4} \epsilon_i = 0, 2, 4$, together with x^N, y^N, z^N, w^N.

Corollary

For $m = (D, \overline{r_1}, \overline{r_2}, \overline{r_3}, \overline{r_4})$ *with* $D = nk + \sum_{i=1}^{4} r_i$*, we have*

$$H_m(R) = H_k\left(\frac{P}{(x^{d+\epsilon_1}, y^{d+\epsilon_2}, z^{d+\epsilon_3}, w^{d+\epsilon_4}) : f}\right)$$

where $f = x + y + z + w$*.*

Corollary

For $m = (D, \overline{r_1}, \overline{r_2}, \overline{r_3}, \overline{r_4})$ *with* $D = nk + \sum_{i=1}^{4} r_i$, *we have*

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where $f = x + y + z + w$.

The left-hand side is the multi-graded Hilbert function and the right-hand side is usual $\mathbf{N}$*-graded Hilbert function.*

The Hilbert function can be explicitely calculated, using the following facts:

- the Hilbert function of the complete intersection

$$C := \frac{P}{(x^{d+\epsilon_1}, y^{d+\epsilon_2}, z^{d+\epsilon_3}, w^{d+\epsilon_4})}$$

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- the map $C_k \xrightarrow{\cdot f} C_{k+1}$ has maximal rank in each degree k, i.e. it is either injective or surjective (the weak Lefschetz Property).
- $$\mathrm{Ker}(\cdot f) = \frac{(x^{d+\epsilon_1}, y^{d+\epsilon_2}, z^{d+\epsilon_3}, w^{d+\epsilon_4}) : f}{(x^{d+\epsilon_1}, y^{d+\epsilon_2}, z^{d+\epsilon_3}, w^{d+\epsilon_4})}$$

Comment:

The Weak Lefschetz Property is the reason why being able to replace $x^n + y^n + z^n + w^n$ *by* $x + y + z + w$ *allows us to perform the calculations for the Hilbert function, as well as the degrees of generators.*

Second simplification

We reduce to a calculation in three variables.

It is enough to find generators of the ideals

$$J_{\underline{\epsilon}} := (x^{d+\epsilon_1}, y^{d+\epsilon_2}, z^{d+\epsilon_3}) : (x+y+z)^{d+\epsilon_4} \subseteq k[x,y,z]$$

for all choices of $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \{0,1\}^4$.

Idea behind this:

Element of $(x^{d_1}, y^{d_2}, z^{d_3}, w^{d_4}) : (x+y+z+w) \quad \rightsquigarrow$
relation on $x^{d_1}, y^{d_2}, z^{d_3}, w^{d^4}, x+y+z+w$

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Element of $(x^{d_1}, y^{d_2}, z^{d_3}, w^{d_4}) : (x+y+z+w) \quad \rightsquigarrow$
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By substituting $w = -(x+y+z)$, this leads to a relation on
$x^{d_1}, y^{d_2}, z^{d_3}, (x+y+z)^{d_4} \quad \rightsquigarrow$
element of $(x^{d_1}, y^{d_2}, z^{d_3}) : (x+y+z)^{d_4}$

For all $d_1, \cdots, d_4$, there is an isomorphism

$$\frac{(x^{d_1}, y^{d_2}, z^{d_3}) : (x+y+z)^{d_4}}{(x^{d_1}, y^{d_2}, z^{d_3})} \xrightarrow{\Phi} \frac{(x^{d_1}, y^{d_2}, z^{d_3}, w^{d}) : (x+y+z+w)}{(x^{d_1}, y^{d_2}, z^{d_3}, w^{d_4})}$$

given by multiplication by

$$P_{d_4} = \frac{w^{d_4} - (-1)^{d_4}(x+y+z)^{d_4}}{x+y+z+w},$$

which raises degrees by $d_4 - 1$.

The ideals $J_{\underline{\epsilon}}$ are **compressed** *Gorenstein ideals of grade 3, with socle degree equal to*

$$2d + \epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4 - 3$$

Compressed: they have maximal Hilbert function among Gorenstein grade three ideals with the give socle degree.

Compressed: they have maximal Hilbert function among Gorenstein grade three ideals with the give socle degree. This follows from the fact that the map

$$\frac{k[x,y,z]}{(x^{d_1},y^{d_2},z^{d_3})} \xrightarrow{\cdot(x+y+z)^{d_4}} \frac{k[x,y,z]}{(x^{d_1},y^{d_2},z^{d_3})}$$

has maximal rank in each degree (so the kernel is as small as possible), i.e.

$$\frac{k[x,y,z]}{(x^{d_1},y^{d_2},z^{d_3})}$$

has the **Strong Lefschetz Property**.

Facts about Compressed ideals

Resolution of compressed Gorenstein ideals have been studied extensively (Boij, Iarrobino, Migliore-Miro-Roig-Nagel), in any nr. of variables.

Depending on the parity of the socle degree, exact betti numbers **or** bounds on the betti numbers are known.

Let J be a compressed grade 3 Gorenstein ideal of grade 3 with socle degree s.

- If s is even, then J is minimally generated by $s+3$ elements of degree $(s/2)+1$, and all the relations are linear.

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- If s is even, then J is minimally generated by $s + 3$ elements of degree $(s/2) + 1$, and all the relations are linear.

- If s is odd, then there are
  - $(s + 3)/2$ generators of degree $(s + 1)/2$
  - $\nu \geq 0$ additional generators in degree $(s + 1)/2 + 1$

  The number ν of additional generators is equal to the number of linear relations in the generators of degree $(s + 1)/2$

The ideals $J_{\underline{\epsilon}} = (x^{d+\epsilon_1}, y^{d+\epsilon_2}, z^{d+\epsilon_3}) : (x+y+z)^{d+\epsilon_4}$ have

- odd socle degree for $\sum_{i=1}^{4} \epsilon_i = 0, 2, 4$
- even socle degree for $\sum_{i=1}^{4} \epsilon_i = 1, 3$

In order to handle the case of odd socle degree,

- we find the required number of explicit elements of the required degree in J_ϵ
- we prove that these elements minimally generate J_ϵ by proving that there are no linear relations.

The result

Ignoring the $\mathbf{G}$-grading, the minimal homogeneous resolution F of R is $F_0 = P$,

$$\begin{aligned}
F_1 &= P(-N)^4 \oplus P(-(2nd-2n+4r))^d \\
&\qquad \oplus P(-(2nd-n+2r))^{6d} \oplus P(-2nd)^{d+1}, \\
F_2 &= P(-(2nd-n+3r))^{8d+4} \oplus P(-(2nd+r))^{8d+4}, \\
F_3 &= P(-(3nd-n+3r))^4 \oplus P(-(2nd+n))^d \\
&\qquad \oplus P(-(2nd+2r))^{6d} \oplus P(-(2nd-n+4r))^{d+1} \\
F_4 &= P(-(4nd-n+4r)).
\end{aligned}$$