Linkage and Tor Algebra Classes of Grade Three Perfect Ideals

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Joint work with
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Free Resolutions and Representation Theory
ICERM, Brown University, Providence, RI
Virtual Presentation
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The talk is based on the following papers

[1] *Linkage classes of grade 3 perfect ideals*,
L.W. Christensen, O. Veliche, J.Weyman,
Journal Pure and Applied Algebra, **224** (2020), no. 6, 106185, 29pp.

[2] *Free resolutions of Dynkin format and the licci property of grade 3 perfect ideals*,
L.W. Christensen, O. Veliche, J. Weyman,
Mathematica Scandinavica, **125** (2019), no. 2, 163 – 178.

Perfect Ideals of Grade 3

Let $(Q, \mathfrak{n}, k)$ be a local ring and I a grade 3 perfect ideal of Q.

A minimal free resolution of Q/I over Q has the form

$$Q \leftarrow Q^m \leftarrow Q^{m+n-1} \leftarrow Q^n \leftarrow 0.$$

We say that I has the resolution format:

$$\mathfrak{f}_I = (1, m, m+n-1, n).$$

m is the number of minimal generators of the ideal I.

n is the type of Q/I, i.e. the rank of the socle $0 :_{Q/I} (\mathfrak{n}/I)$.

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Tor Algebra

Let $(Q, \mathfrak{n}, k)$ be a local ring and I a grade 3 perfect ideal of Q, and let

$$Q/I \leftarrow F_\bullet$$

be a minimal free resolution of Q/I over Q.

Theorem. (Buchsbaum, Eisenbud, 1977) The resolution $F_\bullet$ admits a differential graded algebra structure.

This induces a graded commutative algebra structure on

$$\mathsf{A}_\bullet = \mathsf{H}_\bullet(F_\bullet \otimes_Q k) = \mathrm{Tor}_\bullet^Q(Q/I, k).$$

If I has the format $\mathfrak{f}_I = (1, m, m+n-1, n)$, then $\mathsf{A}_\bullet = k \oplus \mathsf{A}_1 \oplus \mathsf{A}_2 \oplus \mathsf{A}_3$, with

$\mathrm{rank}_k\, \mathsf{A}_1 = m$		$\mathsf{e}_1, \mathsf{e}_2, \ldots, \mathsf{e}_m$
$\mathrm{rank}_k\, \mathsf{A}_2 = m+n-1$	and bases	$\mathsf{f}_1, \mathsf{f}_2, \ldots, \mathsf{f}_{m+n-1}$
$\mathrm{rank}_k\, \mathsf{A}_3 = n$		$\mathsf{g}_1, \mathsf{g}_2, \ldots, \mathsf{g}_n$.

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Classification Theorem

Theorem. (Weyman, 1989; Avramov, Kustin, Miller, 1988) There exist bases $\{e_i\}_{i=1,\ldots,m}$ for A_1, $\{f_j\}_{j=1,\ldots,m+n-1}$ for A_2, and $\{g_\ell\}_{\ell=1,\ldots,n}$ for A_3 such that the non-zero products of the graded commutative algebra $A_\bullet$ are in one of following five classes:

C(3):

$A_1 \cdot A_1$	e_1	e_2	e_3
e_1	0	f_3	$-f_2$
e_2	$-f_3$	0	f_1
e_3	f_2	$-f_1$	0

$A_1 \cdot A_2$	f_1	f_2	f_3
e_1	g_1	0	0
e_2	0	g_1	0
e_3	0	0	g_1

T:

$A_1 \cdot A_1$	e_1	e_2	e_3
e_1	0	f_3	$-f_2$
e_2	$-f_3$	0	f_1
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$A_1 \cdot A_1$	e_1	e_2
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$A_1 \cdot A_2$	f_1	f_2
e_1	g_1	0
e_2	0	g_1

Classification Theorem, Continued

G(r): $r \geq 2$

$A_1 \cdot A_2$	f_1	f_2	$\dots$	f_r
e_1	g_1	0	$\dots$	0
e_2	0	g_1	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
e_r	0	0	$\dots$	g_1

H(p, q): $p \geq 0$, $q \geq 0$

$A_1 \cdot A_1$	e_1	$\dots$	e_p
e_{p+1}	f_1	$\dots$	f_p

$A_1 \cdot A_2$	f_{p+1}	$\dots$	f_{p+q}
e_{p+1}	g_1	$\dots$	g_q

Multiplication Invariants

$$p = \operatorname{rank}_k \mathsf{A}_1 \cdot \mathsf{A}_1$$

$$q = \operatorname{rank}_k \mathsf{A}_1 \cdot \mathsf{A}_2$$

$$\begin{aligned} r = \operatorname{rank}_k(\delta_2^{\mathsf{A}} \colon A_2 &\to \operatorname{Hom}_k(\mathsf{A}_1, \mathsf{A}_3)) \\ f &\mapsto (e \mapsto f \cdot e) \end{aligned}$$

Class of I	p	q	r
C(3)	3	1	3
T	3	0	0
B	1	1	2
G(r), $r \geq 2$	0	1	r
H(p, q)	p	q	q

Multiplication Invariants

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C(3) :

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$A_1 \cdot A_2$	f_1	f_2	f_3
e_1	g_1	0	0
e_2	0	g_1	0
e_3	0	0	g_1

$$\boxed{p = 3, \quad q = 1, \quad r = 3}$$

T :

$A_1 \cdot A_1$	e_1	e_2	e_3
e_1	0	f_3	$-f_2$
e_2	$-f_3$	0	f_1
e_3	f_2	$-f_1$	0

$$A_1 \cdot A_2 = 0$$

$$\boxed{p = 3, \quad q = 0, \quad r = 0}$$

B:

$A_1 \cdot A_1$	e_1	e_2
e_1	0	f_3
e_2	$-f_3$	0

$A_1 \cdot A_2$	f_1	f_2
e_1	g_1	0
e_2	0	g_1

$$\boxed{p = 1, \quad q = 1, \quad r = 2}$$

G(r):
$r \geq 2$

$$A_1 \cdot A_1 = 0$$

$A_1 \cdot A_2$	f_1	f_2	$\ldots$	f_r
e_1	g_1	0	$\ldots$	0
e_2	0	g_1	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
e_r	0	0	$\ldots$	g_1

$$\boxed{p = 0, \quad q = 0, \quad r = r}$$

$\mathbf{H}(0,0)$:

$$A_1 \cdot A_1 = 0 \qquad\qquad A_1 \cdot A_2 = 0$$

$$\boxed{p = 0, \quad q = 0, \quad r = 0}$$

$\mathbf{H}(p,q)$:
$p + q \geq 1$

$A_1 \cdot A_1$	e_1	$\ldots$	e_p
e_{p+1}	f_1	$\ldots$	f_p

$A_1 \cdot A_2$	f_{p+1}	$\ldots$	f_{p+q}
e_{p+1}	g_1	$\ldots$	g_q

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Poincaré Series of the Residue Field

Theorem. (Avramov, 2012) If $(Q, \mathfrak{n}, k)$ is a regular local ring and $I \subseteq \mathfrak{n}^2$ is a grade 3 perfect ideal, then the Poincaré series of the ring Q/I defined by $P_k^{Q/I}(t) = \sum_{i=1}^{\infty} \operatorname{rank}_k \operatorname{Tor}_i^Q(k,k)t^i$ is given by

$$P_k^{Q/I}(t) = \frac{(1+t)^{\operatorname{edim} Q - 1}}{1 - t - (m-1)t^2 - (n-p)t^3 + qt^4 - \tau t^5},$$

where

$$\tau = \begin{cases} 1, & \text{if } I \text{ is of class } \mathbf{C}(3) \text{ or } \mathbf{T} \\ 0, & \text{if } I \text{ is of class } \mathbf{B},\ \mathbf{G}(r),\ \text{or } \mathbf{H}(p,q). \end{cases}$$

Proposition. (Nguyen, — , 2020) If $(Q, \mathfrak{n}, k)$ is a regular local ring and $I \subseteq \mathfrak{n}^2$ is a grade 3, then

$$\tau = \operatorname{rank}_k \operatorname{Coker} \psi,$$

where

$$\psi \colon \mathrm{A}_1 \otimes \mathrm{A}_1 \otimes \mathrm{A}_1 \to (\mathrm{A}_1 \cdot \mathrm{A}_1) \otimes \mathrm{A}_1 \oplus \mathrm{A}_1 \otimes (\mathrm{A}_1 \cdot \mathrm{A}_1)$$

is given by

$$\psi(\mathrm{g} \otimes \mathrm{g}' \otimes \mathrm{g}'') = (\mathrm{g}\mathrm{g}' \otimes \mathrm{g}'', \mathrm{g} \otimes \mathrm{g}'\mathrm{g}''),$$

for all $\mathrm{g}, \mathrm{g}', \mathrm{g}'' \in \mathrm{A}_1$.

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Realizability Question

(Avramov, 2012) Which quintuples

$$(m, n, p, q, r),$$

allowed by the Classification Theorem, are realized by a local ring Q and a perfect ideal I of grade 3?

In particular, which series

$$\frac{(1+t)^{e-1}}{1 - t - (m-1)t^2 - (n-p)t^3 + qt^4 - \tau t^5}$$

is the Poincaré series $P_k^{Q/I}(t)$ of a regular local ring Q of embedding dimension e and a grade 3 perfect ideal I?

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Linkage

Let Q be any local ring and I be a grade 3 perfect ideal.

An ideal $J \subseteq Q$ is said to be directly linked to I if there exists a grade 3 complete intersection ideal $\boldsymbol{x}$ such that $\boldsymbol{x} \subseteq I$ and $J = \boldsymbol{x} : I$.

Theorem. (Golod, 1980) The ideal J is then also a grade 3 perfect ideal with $\boldsymbol{x} \subseteq J$ and $I = \boldsymbol{x} : J$.

An ideal J is said to be linked to I if there exists a sequence of ideals

$$I = J_0, J_1, J_2, \ldots, J_n = J$$

such that J_{i+1} is directly linked to J_i for each $i = 0, \ldots, n-1$. We write $I \sim J$.

The class of the ideal I under this equivalence relation " $\sim$ " is called the linkage class of I.

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Linkage and Tor Algebra Classes

The existence of the five Tor algebra classes was discovered by Weyman using Representation Theory techniques.

The Classification Theorem as stated above was proved by Avramov, Kustin, and Miller using Linkage techniques.

A careful analysis on the change of the Tor algebra class and the resolution format under direct linkage proved to be very fruitful in our research.

Most of our results that follow, are proved using linkage techniques.

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Small values of m and Tor algebra classes

$m = 3$: The ideal I is a complete intersection, hence n=1 and

n	Class of I
1	$\mathbf{C}(3)$

There are no ideals of the resolution format $(1, 3, n + 2, n)$ with $n \geq 2$.

$m = 4$: **Proposition.** (Avramov, 1974; Christensen, – , Weyman, 2020)

n	Class of I
2	$\mathbf{H}(3, 2)$
odd ≥ 3	$\mathbf{T}$
even ≥ 4	$\mathbf{H}(3, 0)$

$m = 5$: **Proposition.** (Christensen, – , Weyman, 2020)

n		Class of I	
1	$\Longleftrightarrow$	$\mathbf{G}(r)$	$r = 5$
2	$\Longleftarrow$	$\mathbf{B}$	
≥ 4	$\Longleftarrow$	$\mathbf{T}$	

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$n = 1$: **Proposition.** (J. Watanabe, 1973) I is a Gorenstein ideal and m is odd, hence

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$n = 2$: **Proposition.** (Brown, 1987) $m \geq 4$ and

m	Class of I
4	$\mathbf{H}(3,2)$
odd ≥ 5	$\mathbf{B}$
even ≥ 6	$\mathbf{H}(1,2)$

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m		Class of I
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≥ 5	$p \neq 3$	not $\mathbf{T}$

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Realizability of Resolution Formats

Theorem. (Christensen, – , Weyman, 2019) Let Q be a local ring of depth at least 3. For every format

$$\mathfrak{f} = (1, m, m+n-1, n) \quad \text{for } m \geq 3 \text{ and } n \geq 1,$$

except for:

$$(1, m, m, 1) \quad \text{for even } m \geq 4 \quad \text{or} \quad (1, 3, n+2, n) \quad \text{for } n \geq 2,$$

there exists a grade 3 perfect ideal in Q of resolution format $\mathfrak{f}$.

Realizability of Class $\mathbf{H}(p, q)$

Consider the resolution format $\mathfrak{f} = (1, m, m + n - 1, n)$. Are there any ideals I of this format and of class $\mathbf{H}(p, q)$?

$\mathbf{H}(p, q)$:
$p \geq 0$
$q \geq 0$

$A_1 \cdot A_1$	e_1	$\dots$	e_p
e_{p+1}	f_1	$\dots$	f_p

$A_1 \cdot A_2$	f_{p+1}	$\dots$	f_{p+q}
e_{p+1}	g_1	$\dots$	g_q

It is clear that the following inequalities hold:

$$p \leq m - 1 \quad \text{and} \quad q \leq n.$$

The next result gives necessary conditions on p and q for such ideals to exist.

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$A_1 \cdot A_1$	e_1	$\dots$	e_p
e_{p+1}	f_1	$\dots$	f_p

$A_1 \cdot A_2$	f_{p+1}	$\dots$	f_{p+q}
e_{p+1}	g_1	$\dots$	g_q

It is clear that the following inequalities hold:

$$p \leq m-1 \quad \text{and} \quad q \leq n.$$

The next result gives necessary conditions on p and q for such ideals to exist.

Realizability of Class $\mathbf{H}(p,q)$

Consider the resolution format $\mathfrak{f} = (1, m, m+n-1, n)$. Are there any ideals I of this format and of class $\mathbf{H}(p,q)$?

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Then the following inequalities hold

$$p \leq m-1 \quad \text{and} \quad q \leq n,$$

and the following conditions are equivalent

$$(i)\ p = n+1 \qquad (ii)\ q = m-2 \qquad (iii)\ p = m-1 \quad \text{and} \quad q = n.$$

Otherwise, i.e. when these conditions are not satisfied, there are inequalities

$$p \leq n-1 \quad \text{and} \quad q \leq m-4$$

with

$$p = n-1 \ \text{ only if } \ q \equiv_2 m-4 \quad \text{and} \quad q = m-4 \ \text{ only if } \ p \equiv_2 n-1\,.$$

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Illustration in Case $m = 7$ and $n = 5$

$$0 \leq p \leq m - 1 \quad \text{and} \quad 0 \leq q \leq n$$

$\mathbf{H}(0,5)$	$\mathbf{H}(1,5)$	$\mathbf{H}(2,5)$	$\mathbf{H}(3,5)$	$\mathbf{H}(4,5)$	$\mathbf{H}(5,5)$	$\mathbf{H}(6,5)$
$\mathbf{H}(0,4)$	$\mathbf{H}(1,4)$	$\mathbf{H}(2,4)$	$\mathbf{H}(3,4)$	$\mathbf{H}(4,4)$	$\mathbf{H}(5,4)$	$\mathbf{H}(6,4)$
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$$p = n - 1 \quad \text{only if} \quad q \equiv_2 m - 4 \qquad \text{and} \qquad q = m - 4 \quad \text{only if} \quad p \equiv_2 n - 1 \; .$$

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Open Problem on Realizability of Class $\mathbf{H}(p, q)$

Given a local ring Q, a resolution format $\mathfrak{f} = (1, m, m+n-1, n)$ for $m \geq 4$ and $n \geq 2$, and non-zero integers p and q satisfying one of the following conditions:

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are there any grade 3 perfect ideals of format $\mathfrak{f}$ and of class $\mathbf{H}(p, q)$?

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Realizability of Class $\mathbf{G}(r)$

Remark. Let I be perfect grade 3 ideal of resolution format $\mathfrak{f}_I = (1, m, m + n - 1, n)$. Then I is Gorenstein, not complete intersection, if and only if I is of class $\mathbf{G}(m)$; in this case necessarily m is odd, $m \geq 5$, and $n = 1$.

Theorem. (Christensen, –, Weyman, 2014, 2015) Let Q be the power series algebra in three variables over a field. For every $r \geq 2$ there is quotient ring of Q that is of class $\mathbf{G}(r)$ and not Gorenstein.

Theorem. (Avramov, 2012) If I is a perfect ideal of grade 3 that of class $\mathbf{G}(r)$ that is not Gorenstein, then

$$2 \leq r \leq m - 2.$$

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Conjecture on Realizability of Class $\mathbf{G}(r)$

If I is a grade 3 perfect ideal of class $\mathbf{G}(r)$, not Gorenstein, then the following hold:

If $n = 2$, then one has

$$2 \leq r \leq m - 5 \quad \text{or} \quad r = m - 3.$$

If $n \geq 3$, then one has

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Proposition. (Christensen, –, Weyman, 2020) If I is a grade 3 perfect ideal of Q of class $\mathbf{G}(r)$, not Gorenstein, then $m \geq 6$.

Moreover, if $m = 6$ and $n \geq 3$, then $r = 2$.

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Tor Algebra of Perfect Ideals of Grade at Most 3

Assume that Q is a *regular* local ring and I a perfect ideal with $\operatorname{grade} I \leq 3$.

grade I	name of Q/I	class of I	
0	Regular	$\mathbf{C}(0)$	
1	Hypersurface	$\mathbf{C}(1)$	
2	Complete intersection	$\mathbf{C}(2)$	
	Golod	$\mathbf{S}$	
3	Complete intersection	$\mathbf{C}(3)$	
	Gorenstein, not c.i.	$\mathbf{G}(m)$	odd $m \geq 5$
	Golod	$\mathbf{H}(0,0)$	
	Truncated c.i.	$\mathbf{T}$	
	Brown	$\mathbf{B}$	
	-	$\mathbf{G}(r)$	$r \leq m-2$
	-	$\mathbf{H}(p,q)$	$p+q \geq 1$

Linkage Classes of Grade 3 Perfect Ideals

A perfect ideal I is said to be licci if it is in the linkage class of a complete intersection ideal, i.e. if there exists a sequence of ideals $I = J_0, J_1, J_2, \ldots, J_n = J$ such that J_{i+1} is directly linked to J_i for each $i = 0, \ldots, n-1$ and J is a complete intersection.

Every grade 2 perfect ideal is licci, but not every grade 3 perfect ideal is licci.

Theorem. (Christensen, – , Weyman, 2020) Every grade 3 perfect ideal is linked to a grade 3 perfect ideal of class

$$\mathbf{C}(3) \quad \text{or} \quad \mathbf{H}(0,0).$$

In particular, if Q is a regular ring, then every perfect ideal of grade 3 is licci or in the linkage class of a Golod ideal.

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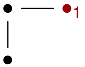
Resolution Formats and Dynkin Diagrams I

$(1, 3, 3, 1)$



A_3

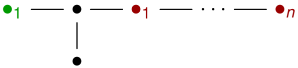
For odd $m \geq 5$

$(1, m, m, 1)$

D_m

For $n \geq 2$

$(1, 4, n+3, n)$



D_{n+3}

Resolution Formats and Dynkin Diagrams II

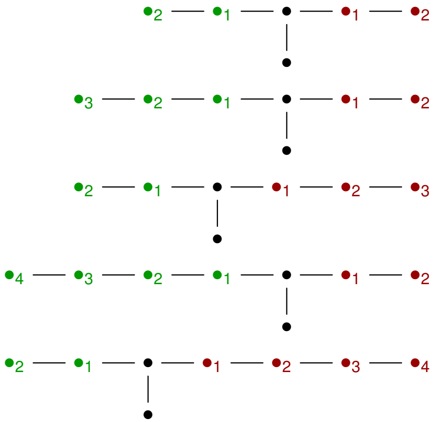
(1, 5, 6, 2)

(1, 6, 7, 2)

(1, 5, 7, 3)

(1, 7, 8, 2)

(1, 5, 8, 4)



E_6

E_7

E_7

E_8

E_8

Licci Conjecture

Let Q be a regular local ring and

$$\mathfrak{f} = (1, m, m+n-1, n)$$

be a resolution format realized by some grade 3 perfect ideal in Q.

I If $\mathfrak{f}$ is not Dynkin, there exists a grade 3 perfect ideal of format $\mathfrak{f}$ that is not licci.

II If $\mathfrak{f}$ is Dynkin, then every grade 3 perfect ideal of format $\mathfrak{f}$ is licci.

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Evidence for Conjecture I

Theorem. (Christensen, – , Weyman, 2019) Let $\Bbbk$ a field, $e \geq 3$, and

$$Q = \Bbbk[X_1, \ldots, X_e]_{(X_1,\ldots,X_e)}.$$

For every non Dynkin format

$$\mathfrak{f} = (1, m, m+n-1, n) \quad \text{with } m \geq 3 \text{ and } n \geq 1$$

there exists a grade 3 perfect ideal that has resolution format $\mathfrak{f}$ and is not licci.

Proposition. (Christensen, – , Weyman, 2019) Let Q be a local ring. If there is a resolution format $\mathfrak{f}$ such that

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3. every perfect ideal in Q of format $\mathfrak{f}$ is licci

then $(1, 6, 8, 3)$ or $(1, 8, 9, 2)$ is such a format.

Theorem. (Huneke, Ulrich, 1987) Let $\mathbb{k}$ be a field and $e \geq 3$ and set $\mathcal{Q} = \mathbb{k}[X_1, \ldots, X_e]$.

Let $\mathcal{I}$ be a homogeneous perfect ideal in $\mathcal{Q}$ of grade 3 with minimal free resolution

$$\mathcal{Q} \longleftarrow \bigoplus_{i=1}^{m} \mathcal{Q}(-d_{1,i}) \longleftarrow \bigoplus_{i=1}^{m+n-1} \mathcal{Q}(-d_{2,i}) \longleftarrow \bigoplus_{i=1}^{n} \mathcal{Q}(-d_{3,i}) \longleftarrow 0 \,.$$

where $d_{1,1} \leq d_{1,2} \leq \cdots \leq d_{1,m}$ and $d_{3,1} \leq \cdots \leq d_{3,n}$.

Set $Q = \mathcal{Q}_{(X_1,\ldots,X_e)}$ and $I = \mathcal{I}_{(X_1,\ldots,X_e)}$.

If the inequality

$$d_{3,n} \leq 2d_{1,1}$$

holds, then the ideal I is not licci.

Non-licci ideals of formats

$(1, 6, 8, 3)$ or $(1, 8, 9, 2)$

Let $\Bbbk$ be a field and set $\mathcal{Q} = \Bbbk[X, Y, Z]$.

Example 1. The ideal $\mathcal{I}$ generated by the 2×2 minors of the matrix

$$\begin{pmatrix} X & Y & Z & 0 \\ 0 & X & Y & Z \end{pmatrix}$$

has the minimal free resolution over $\mathcal{Q}$ given by:

$$\mathcal{Q} \longleftarrow \mathcal{Q}(-2)^6 \longleftarrow \mathcal{Q}(-3)^8 \longleftarrow \mathcal{Q}(-4)^3 \longleftarrow 0\,.$$

Example 2. The ideal

$$\mathcal{I} = (X^3, X^2Y + YZ^2, X^2Z + XYZ, XY^2 + XYZ, XZ^2, Y^3, Y^2Z, Z^3)$$

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Evidence for Conjecture II

Theorem. Let $\Bbbk$ be a field and $e \geq 3$ and set $\mathcal{Q} = \Bbbk[X_1, \ldots, X_e]$.
Let $\mathcal{I}$ be a homogeneous perfect ideal in $\mathcal{Q}$ of grade 3 with initial degree $d_{1,1} \geq 2$ and minimal free resolution

$$\mathcal{Q} \longleftarrow \bigoplus_{i=1}^{m} \mathcal{Q}(-d_{1,i}) \longleftarrow \bigoplus_{i=1}^{m+n-1} \mathcal{Q}(-d_{2,i}) \longleftarrow \bigoplus_{i=1}^{n} \mathcal{Q}(-d_{3,i}) \longleftarrow 0\,.$$

(a) If the ideal $\mathcal{I}$ has resolution format

$$(1, m, m+1, 2) \quad \text{for } 4 \leq m \leq 7,$$

then one has

$$d_{3,2} > d_{1,1} + d_{1,2} \geq 2d_{1,1}.$$

(b) If the ideal $\mathcal{I}$ has resolution format

$$(1, 5, n+4, n) \quad \text{for } 1 \leq n \leq 4,$$

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