The Family of Grade 3 Perfect Ideals of Type 2 with 5 Generators

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Free Resolutions and Representation Theory
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Notation/Definition

Let R be a commutative ring.

- Let $\mathbb{F}_\bullet : 0 \to F_n \xrightarrow{d_n} F_{n-1} \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0$ be a complex of free R-modules.
- r_i :=rank(d_i) is the size of the biggest non-vanishing minor of d_i.
- $\underline{r} = (r_1, \ldots, r_n) :=$ the rank sequence of $\mathbb{F}_\bullet$
- $\mathbb{F}_\bullet$ is of format $\underline{f} = (f_0, \ldots, f_n)$ if $\text{rank}(F_i) = f_i$ for each i.
- $\text{rank}(F_i) = r_i + r_{i+1}$ $(1 \leq i \leq n)$ with $r_{n+1} = 0$ by the Buchsbaum-Eisenbud Exactness Criterion. WLOG assume $\text{rank}(F_0) = r_1$.

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- An ideal I in a commutative Noetherian ring R is said to be perfect if $\text{grade}_R(I) = \text{projdim}_R(R/I)$.
- If R is local, I is perfect of grade n, and $\mathbb{F}$ is a minimal free resolution of R/I, then $\text{type}(I) = \text{rank}(F_n)$.

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- **Hilbert-Burch Theorem, 1968**

  If R is a local ring with an ideal I and

  $$0 \to R^m \xrightarrow{\varphi} R^n \to R \to R/I \to 0$$

  is a free resolution of R/I, then $m = n - 1$ and there exists a regular element a of R such that $I = aI_m(\varphi)$, where $I_m(\varphi)$ is an ideal generated by $m \times m$ minors of φ.

Motivation

- **First Structure Theorem (Buchsbaum-Eisenbud, 1974)**

  Let $\mathbb{F}$ be a finite free resolution over a Noetherian ring R:

  $$\mathbb{F} : 0 \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0$$

  Then there exist unique maps $a_i : R \to \bigwedge^{r_i} F_{i-1}$ such that

  - $a_n = \bigwedge^{r_n} d_n$,
  - We have factorizations $\bigwedge^{r_i} F_i = \bigwedge^{r_{i+1}} F_i^* \xrightarrow{\wedge^{r_i} d_i} \bigwedge^{r_i} F_{i-1}$

  $$\begin{array}{ccc} \bigwedge^{r_{i+1}} F_i^* & \xrightarrow{\wedge^{r_i} d_i} & \bigwedge^{r_i} F_{i-1} \\ & \searrow^{a_{i+1}^*} & \uparrow{a_i} \\ & & R \end{array}$$

  We refer to the maps a_i as Buchsbaum-Eisenbud multipliers.

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- 2020: Christensen-Veliche-Weyman recovered the form of the almost complete intersection of grade 3 from the generic ring and realized it is a D_n Schubert variety.

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  This classification comes with the additional assumption that one of the Koszul relations on the generators is also one of the minimal generators of the first syzygy module of the ideal. In that case, the ideals are licci.

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  This classification comes with the additional assumption that one of the Koszul relations on the generators is also one of the minimal generators of the first syzygy module of the ideal. In that case, the ideals are licci.
- Applying minimal linkage to a grade 3 perfect ideal gives an ideal with a minimal resolution with the same sum of Betti numbers as the original one.
- After double linkage, the ideal we get has the free resolution with modules of the same ranks as for the original ideal.

Generic Free Resolutions

- Fix a format $\underline{f} = (f_0, \ldots, f_n)$ and the rank sequence $\underline{r} = (r_1, \ldots, r_n)$. Consider the pairs $(R, \mathbb{F}_\bullet)$ with R Noetherian, $\mathbb{F}_\bullet$ free acyclic of the given format and rank sequence.

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- The pair $(R_{gen}, \mathbb{F}_\bullet^{gen})$ is **generic** for format $\underline{f}$ and rank sequence $\underline{r}$ if

  (1) $\mathbb{F}_\bullet^{gen}$ is acyclic,

  (2) For every pair $(S, \mathbb{G}_\bullet)$ of the same format and rank, $\exists$ a ring map (not necessarily unique) $\varphi : R_{gen} \to S$ such that $\mathbb{G}_\bullet = \mathbb{F}_\bullet^{gen} \otimes_{R_{gen}} S$.

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- **Bruns - 1984**: For every format $(f_0, \cdots, f_n)$, $\exists$ a generic pair (R_{gen}, F_{gen}) in which R_{gen} is not always Noetherian.
- One wants to find R_{gen} explicitly and determine if it is Noetherian.

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Recall: rank $F_i = f_i$ $(0 \leq i \leq 3)$ and if rank $(d_i) = r_i$ $(1 \leq i \leq 3)$, then it follows $f_3 = r_3$, $f_2 = r_3 + r_2$, $f_1 = r_2 + r_1$.

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$$(f_0, f_1, f_2, f_3) = (r_1, r_1 + r_2, r_2 + r_3, r_3).$$

  - This generic ring $\hat{R}_{gen}$ was related to Kac-Moody Lie algebra corresp. to the graph $T_{p,q,r}$, where $(p, q, r) = (r_1 + 1, r_2 - 1, r_3 + 1)$.

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  - In particular, the generic ring $\hat{R}_{gen}$ is Noetherian if and only if $T_{p,q,r}$ is a Dynkin graph, i.e., a graph of type A_n, D_n, E_6, E_7, E_8.

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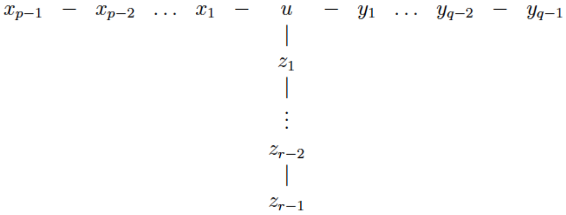
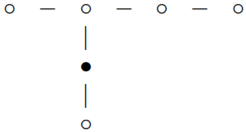
- The graph $T_{p,q,r}$ corresponds to the graph of type $E_6 \Rightarrow$ $\hat{R}_{gen}$ is Noetherian. Note $p=2, q=3, r=3$ here.





T-shaped graph $T_{p,q,r}$

$T_{2,3,3}$: Type E_6

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- The family we got comes from picking a certain slice in the spectrum of the ring $\hat{R}_{gen}$.
- The ideals that I will introduce (J and $J(t)$) are linear sections for Schubert variety of codimension 3 in $G(E_6)/P_2$ where P_2 is the maximal parabolic corresponding to simple root α_2.

  The variable t actually corresponds to the highest root of E_6.

Example: The Family of Grade 3 Perfect Ideals of Type 2 with 5 Generators

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- Let K be a field of characteristic different from two.
- Let F and G be two vectors spaces over K of dimensions 4 and 2.
- Consider the affine space

$$X = \bigwedge^2 F^* \otimes G^* \oplus \bigwedge^3 F^*$$

  whose elements are pairs consisting of a pencil of 4×4 skew-symmetric matrices and a 4-vector.
- Then the coordinate ring A of X can be identified with the symmetric algebra

$$A = \mathrm{Sym}(\bigwedge^2 F \otimes G \oplus \bigwedge^3 F).$$

Example – Continued

- Denote the coordinate functions in the coordinate ring A by $x_{i,j}$, $y_{i,j}$ $(1 \leq i < j \leq 4)$, and $z_{i,j,k}$ $(1 \leq i < j < k <\leq 4)$.
- A is a bigraded ring with $\deg(x_{i,j}) = \deg(y_{i,j}) = (1,0)$ and $\deg(z_{i,j,k}) = (0,1)$.

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- A is a bigraded ring with $\deg(x_{i,j}) = \deg(y_{i,j}) = (1,0)$ and $\deg(z_{i,j,k}) = (0,1)$.
- Now consider the equivariant ideal J in A generated by the representation $S_{2,2,2,1}F \otimes \bigwedge^2 G$ in bidegree $(2,1)$ and the representation $S_{2,2,2,2}F \otimes S_{2,2}G$ in bidegree $(4,0)$.

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- We describe the generators of the ideal J explicitly. For this we use $\Delta(ij, kl)$ to denote the 2×2 minor of the matrix

$$\begin{bmatrix} x_{1,2} & x_{1,3} & x_{1,4} & x_{2,3} & x_{2,4} & x_{3,4} \\ y_{1,2} & y_{1,3} & y_{1,4} & y_{2,3} & y_{2,4} & y_{3,4} \end{bmatrix}$$

corresponding to the columns labeled by (i, j) and (k, l).

Example – Continued

- The cubic generators of bidegree $(2,1) = \{u_{1,2,3},\, u_{1,2,4},\, u_{1,3,4},\; u_{2,3,4}\}$:

$$\begin{aligned}
u_{1,2,3} = & -2z_{2,3,4}\Delta(12,13) + 2z_{1,3,4}\Delta(12,23) - 2z_{1,2,4}\Delta(13,23) \\
& + z_{1,2,3}(\Delta(13,24) - \Delta(12,34) + \Delta(14,23)) \\
u_{1,2,4} = & \, 2z_{2,3,4}\Delta(12,14) - 2z_{1,3,4}\Delta(12,24) + z_{1,2,4}(\Delta(12,34) + \Delta(13,24) + \Delta(14,23)) \\
& - 2z_{1,2,3}\Delta(14,24) \\
u_{1,3,4} = & \, 2z_{2,3,4}\Delta(13,14) + z_{1,3,4}(-\Delta(12,34) - \Delta(13,24) + \Delta(14,23)) + 2z_{1,2,4}\Delta(13,34) \\
& - 2z_{1,2,3}\Delta(14,34) \\
u_{2,3,4} = & \, z_{2,3,4}(-\Delta(12,34) - \Delta(13,24) - \Delta(14,23)) - 2z_{1,3,4}\Delta(23,24) + 2z_{1,2,4}\Delta(23,34) \\
& - 2z_{1,2,3}\Delta(24,34).
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Example – Continued

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\end{aligned}$$

- The generator of degree $(4,0)$ is a discriminant of the Pfaffian of the skew-symmetric 4×4 matrix of linear forms treated as a binary form which is equal to $u = b^2 - 4ac$ where

$$\begin{aligned}
a &= x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3}, \\
b &= x_{1,2}y_{3,4} - x_{1,3}y_{2,4} + x_{1,4}y_{2,3} + x_{3,4}y_{1,2} - x_{2,4}y_{1,3} + x_{2,3}y_{1,4}, \\
c &= y_{1,2}y_{3,4} - y_{1,3}y_{2,4} + y_{1,4}y_{2,3}.
\end{aligned}$$

Example – Continued

- Then the minimal resolution of A/J over A is

$$\mathbb{F}_\bullet : 0 \to A^2 \xrightarrow{d_3} A^6 \xrightarrow{d_2} A^5 \xrightarrow{d_1} A \ \text{ where}$$

$$d_1 = \begin{bmatrix} u_{2,3,4} & u_{1,3,4} & u_{1,2,4} & u_{1,2,3} & u \end{bmatrix},$$

$$d_2 = \begin{bmatrix} v_1 & u_1 & -\delta_1+\delta_2-\delta_3 & 2\Delta(13,14) & -2\Delta(12,14) & -2\Delta(12,13) \\ -v_2 & -u_2 & -2\Delta(23,24) & -\delta_1-\delta_2+\delta_3 & 2\Delta(12,24) & 2\Delta(12,23) \\ v_3 & u_3 & 2\Delta(23,34) & 2\Delta(13,34) & -\delta_1-\delta_2-\delta_3 & -2\Delta(13,23) \\ v_4 & u_4 & 2\Delta(24,34) & 2\Delta(14,34) & -2\Delta(14,24) & \delta_1-\delta_2-\delta_3 \\ 0 & 0 & -z_{2,3,4} & -z_{1,3,4} & z_{1,2,4} & z_{1,2,3} \end{bmatrix},$$

$$d_3 = \begin{bmatrix} b & 2a \\ -2c & -b \\ -v_1 & -u_1 \\ v_2 & u_2 \\ v_3 & u_3 \\ -v_4 & -u_4 \end{bmatrix}.$$

and $u_j = \sum_{i\neq j}(-1)^i x_{i,j} z_{\hat{i}}$, $v_j = \sum_{i\neq j}(-1)^i y_{i,j} z_{\hat{i}}$, $\delta_1 = \Delta(12,34)$, $\delta_2 = \Delta(13,24)$, and $\delta_3 = \Delta(14,23)$.

The Deformed Ideal $J(t)$

- We now give a deformed ideal $J(t)$ which is an ideal in the bigger polynomial ring $B = A[t]$.
- Equivariantly, the variable t is $\bigwedge^4 F \otimes \bigwedge^2 G$. Set $\deg(t) = 2$.
- Then the matrices of differentials d_2 and d_3 become

$$d_2(t) = \begin{bmatrix} v_1 & u_1 & -\delta_1 + \delta_2 - \delta_3 + t & 2\Delta(13,14) & -2\Delta(12,14) & -2\Delta(12,13) \\ -v_2 & -u_2 & -2\Delta(23,24) & -\delta_1 - \delta_2 + \delta_3 + t & 2\Delta(12,24) & 2\Delta(12,23) \\ v_3 & u_3 & 2\Delta(23,34) & 2\Delta(13,34) & -\delta_1 - \delta_2 - \delta_3 - t & -2\Delta(13,23) \\ v_4 & u_4 & 2\Delta(24,34) & 2\Delta(14,34) & -2\Delta(14,24) & \delta_1 - \delta_2 - \delta_3 + t \\ 0 & 0 & -z_{2,3,4} & -z_{1,3,4} & z_{1,2,4} & z_{1,2,3} \end{bmatrix},$$

$$d_3(t) = \begin{bmatrix} b+t & 2a \\ -2c & -b+t \\ -v_1 & -u_1 \\ v_2 & u_2 \\ v_3 & u_3 \\ -v_4 & -u_4 \end{bmatrix},$$

where $u_j = \sum_{i \neq j} (-1)^i x_{ij} z_{\hat{i}}$, $v_j = \sum_{i \neq j} (-1)^i y_{ij} z_{\hat{i}}$, $\delta_1 = \Delta(12,34)$, $\delta_2 = \Delta(13,24)$, and $\delta_3 = \Delta(14,23)$.

The Deformed Ideal $J(t)$

- Then, $\mathrm{im}[d_1(t)]^T = \ker[d_2(t)]^T$, where $[d_2(t)]^T$ and $[d_1(t)]^T$ are transposes of matrices $d_2(t)$ and $d_1(t)$, respectively.
- Thus, we have

$$\begin{aligned} J(t) &= \mathrm{im}(d_1(t)) \\ &= \langle -u_{2,3,4} + tz_{2,3,4}, -u_{1,3,4} + tz_{1,3,4}, -u_{1,2,4} + tz_{1,2,4}, u_{1,2,3} - tz_{1,2,3}, u - t^2 \rangle. \end{aligned}$$

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Theorem (–, J. Laxmi, W. Kraśkiewicz, J. Weyman)

The ideal $J(t)$ is a perfect ideal of codimension three in B. The minimal free resolution of $B/J(t)$ over B is

$$0 \to B^2(-7) \xrightarrow{d_3(t)} B^6(-5) \xrightarrow{d_2(t)} B(-4) \oplus B^4(-3) \xrightarrow{d_1(t)} B$$

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The ideals J and $J(t)$ are in the linkage class of complete intersections.

Proof of the Example (Sketch):

- We use the Buchsbaum-Eisenbud exactness criterion:

  Buchsbaum-Eisenbud Exactness Criterion: The complex

$$\mathbb{F}_\bullet : 0 \to F_n \xrightarrow{d_n} F_{n-1} \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0$$

  over a Noetherian ring R is acyclic if and only if

  - $r_i + r_{i+1} = \text{rank}(F_i) \quad \forall\ i = 1, \cdots, n,$
  - $\text{depth}_R(I(d_i)) \geq i$ for $i = 1, \cdots, n$, where $I(d_i)$ denotes the ideal generated by $r_i \times r_i$ minors of d_i.

- One can easily check that the rank conditions are satisfied.

- Thus we need to show that for $1 \leq i \leq 3$, $\text{depth}_R(I(d_i)) \geq 3$.

Sketch of the Proof – Continued:

- Enough to show that the codimension of the ideal J is 3 and that all the ideals $I(d_i)$ for $1 \leq i \leq 3$ are the same up to radical.

- To prove this, we used a geometric interpretation of the ideal J.
- The element u is the hyperdiscriminant for the representation $\bigwedge^2 F^* \otimes G^*$.

- The zero set $V(J)$ consists of pairs $((x_{i,j}, y_{i,j}), (z_{i,j,k}))$ such that the trivector $(z_{i,j,k})$ viewed as a functional on F^* vanishes on the kernel of the rank two member of the pencil.

- The resolution with differentials d_3, d_2, d_1 can be constructed by a geometric method.

- It turns out that the variety $V(J)$ has rational singularities so A/J is Cohen-Macaulay of codimension 3.

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Thank You!

Recall: Deformation of (R, I) and Linkage

Let (R, I) and (S, J) be pairs, where R, S are Noetherian rings, and $I \subseteq R$, $J \subseteq S$ are ideals.

- (R, I) and (S, J) are said to be isomorphic, i.e., $(R, I) \cong (S, I)$, if there is an isomorphism $\phi : R \to S$ with $\phi(I) = J$.
- (S, J) is called a deformation of (R, I), if there is a sequence $\mathbf{a} \subseteq S$, $\mathbf{a} = a_1, \ldots, a_n$, which is regular on S and S/J such that $(S/(\mathbf{a}), (J + \mathbf{a})/(\mathbf{a}) \cong (R, I)$

Let I be a perfect ideal of grade 3 in a local ring R.

- An ideal $J \subseteq I$ said to be directly linked to I if there exists a regular sequence x_1, x_2, x_3 contained in I with $J = (x_1, x_2, x_3) :_R I$.
- The ideal J is then also a perfect ideal of grade 3. Note if J is directly linked to I, then I is also directly linked to J.
- An ideal J is said to be linked to I if $\exists$ a sequence of ideals $I = J_0, J_1, \cdots, J_n = J$ such that I is directly linked to J_i for $i = 0, \ldots, n-1$.