Algebraic Methods for Tensor Data

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Tensors

$V_i = \mathbb{R}^{p_i}$
$V = V_1 \otimes V_2 \otimes \cdots \otimes V_d = \mathbb{R}^{p_1 \times p_2 \times \cdots \times p_d}$
$\mathcal{T} \in V$ tensor

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$\mathcal{T} \cdot \mathcal{S}$ inner product of tensors
$\|\mathcal{T}\| = |\mathcal{T}\|_F = \sqrt{\mathcal{T} \cdot \mathcal{T}}$ Euclidean (Frobenius) norm

Tensor Rank

Definition

rank($\mathcal{T}$) is the smallest integer r for which we have a decomposition

$$(\star) \qquad \mathcal{T} = \sum_{j=1}^{r} v_{1j} \otimes v_{2j} \otimes \cdots \otimes v_{dj}$$

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tensor rank is not continuous or even semi-continuous, i.e., $\{\mathcal{T} \in V \mid \text{rank}(\mathcal{T}) \leq r\}$ not always a closed set

Nuclear Norm

use convex relaxation of tensor rank:

Definition

the nuclear norm $\|\mathcal{T}\|_\star$ of $\mathcal{T}$ is the minimal value of

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where r is arbitrary and

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(a minimum is attained)

if $\|\mathcal{T}\|_\star$ is equal to $(\#)$, then $(\star)$ is called the nuclear decomposition or convex decomposition

Spectral Norm

Definition

the spectral norm of $\mathcal{T}$ is

$$\|\mathcal{T}\|_\sigma = \max\{\mathcal{T}\cdot(v_1 \otimes v_2 \otimes \cdots \otimes v_d) \mid \|v_1\| = \|v_2\| = \cdots = \|v_d\| = 1\}$$

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$\|\cdot\|_\sigma$ and $\|\cdot\|_\star$ are dual norms

$$|\mathcal{T}\cdot\mathcal{S}| \leq \|\mathcal{T}\|_\star\|\mathcal{S}\|_\sigma$$

Matrices ($d = 2$)

$A \in \mathbb{R}^{p \times q}$ has a singular value decomposition
$A = UDV^t$ with $U \in \mathrm{O}_p$, $V \in \mathrm{O}_q$, D diagonal with the nonzero diagonal entries $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$

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the norms and the rank of A are easy to compute

negative results ($d = 3$)

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computing tensor rank is NP-complete

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Theorem (Hillar–Lim)

computing the spectral norm is NP-complete

Approximating Spectral Norm

(case $d = 3$ for simplicity)

$$\|\mathcal{T}\|_{\overline{\sigma},d} = \left(\int_{S^{p-1} \times S^{q-1} \times S^{r-1}} |\mathcal{T} \cdot (x \otimes y \otimes z)|^d \right)^{\frac{1}{d}} .$$

normalize

$$\|\mathcal{T}\|_{\sigma,d} = \frac{\|\mathcal{T}\|_{\overline{\sigma},d}}{\|e_1 \otimes e_1 \otimes e_1\|_{\overline{\sigma},d}}$$

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when d is even there is an algebraic method for computing $\|\mathcal{T}\|_{\sigma,d}$!

Invariant Tensors

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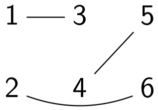
there is an O_n-equivariant linear isomorphism between $V^{\otimes d}$ and the space of multilinear maps $V^d \to \mathbb{R}$

so there is a linear isomorphism between the space $(V^{\otimes d})^{\mathrm{O}_n}$ of O_n-invariant tensors and the space of O_n-invariant multilinear maps $V^d \to \mathbb{R}$

Brauer Diagrams

a Brauer diagram is a perfect matching, for example

D = 

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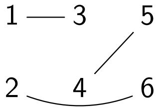

to a diagram E on d vertices we can associate an O_n-invariant multilinear map $\mathcal{M}_E : V^d \to \mathbb{R}$, for example

$$\mathcal{M}_D(v_1, v_2, \ldots, v_6) = (v_1 \cdot v_3)(v_2 \cdot v_6)(v_4 \cdot v_5)$$

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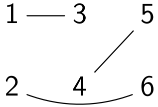

to a diagram E on d vertices we can associate an O_n-invariant multilinear map $\mathcal{M}_{\mathrm{E}} : V^d \to \mathbb{R}$, for example

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the invariant multi-linear map $\mathcal{M}_{\mathrm{E}}$ corresponds to an invariant tensor $\mathcal{T}_{\mathrm{E}}$ using the linear isomorphism of before, for example

$$\mathcal{T}_{\mathrm{D}} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} e_i \otimes e_j \otimes e_i \otimes e_k \otimes e_k \otimes e_j$$

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Theorem (FFT of Invariant Theory for O_n)

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the number of Brauer diagrams on d vertices is $1 \cdot 3 \cdot 5 \cdots d - 1$ (when d even)

Partial Brauer Diagrams

to a partial Brauer diagram with d vertices and e edges we associate an $\mathrm{O}(V)$-equivariant multi-linear map $\mathcal{M}_\mathrm{D} : V^d \to V^{\otimes(d-2e)}$ and a linear map $\mathcal{L}_\mathrm{D} : V^{\otimes d} \to V^{\otimes d-2e}$

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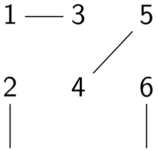
for example, if

D = 

then

$$\mathcal{M}_{\mathrm{D}}(v_1, v_2, \ldots, v_6) = (v_1 \cdot v_3)(v_4 \cdot v_5) v_2 \otimes v_6 \in V^{\otimes 2}$$

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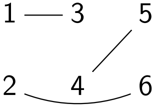
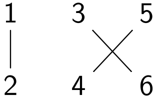
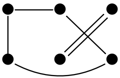
D = 

E = 

then

$$\mathcal{T}_{\mathsf{D}} \cdot \mathcal{T}_{\mathsf{E}} = n^2 \qquad \text{because}$$



has 2 cycles

Integrating

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from

$$1 = \int_{S^{n-1}} 1 = \int_{S^{n-1}} v^{\otimes d} \cdot \mathcal{T}_{\mathrm{E}} = C\mathcal{S}_d \cdot \mathcal{T}_{\mathrm{E}} = Cn(n+2)\cdots(n+d-2)$$

follows $C = \frac{1}{n(n+2)\cdots(n+d-2)}$

Colored Brauer Diagrams

$R = \mathbb{R}^p$, $G = \mathbb{R}^q$, $B = \mathbb{R}^r$
$V = R \otimes G \otimes B$ space of order 3 tensors
$\psi : R^{\otimes d} \otimes G^{\otimes d} \otimes B^{\otimes d} \to V^{\otimes d}$ linear isomorphism
$H = O_p \times O_q \times O_r$ acts on V, $V^{\otimes d}$

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$(V^{\otimes d})^H$ spanned by all $\psi(\mathcal{T}_{\mathrm{D}_R} \otimes \mathcal{T}_{\mathrm{D}_G} \otimes \mathcal{T}_{\mathrm{D}_B})$ where $\mathrm{D}_R, \mathrm{D}_G, \mathrm{D}_B$ are Brauer diagrams on d vertices

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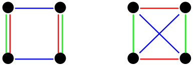
we draw the Brauer diagrams D_R in red, D_G in green, and D_B in blue using the same d vertices and we get a *colored Brauer diagram* D, for example



Tensor Invariants

$V = R \otimes G \otimes B$, space of $p \times q \times r$ tensors

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for a colored Brauer diagram, define a polynomial function $\mathcal{I}_{\mathcal{D}} : V \to \mathbb{R}$ by $\mathcal{I}_{\mathcal{D}}(\mathcal{U}) = \mathcal{U}^{\otimes d} \cdot \mathcal{T}_{\mathrm{D}}$

Theorem

the ring of H-invariant polynomials on the space of $p \times q \times r$ tensors is spanned by all $\mathcal{I}_{\mathrm{D}}$, where D is a colored Brauer diagram

Computing the Integral

Up to a constant

$$\int_{S^{p-1}\times S^{q-1}\times S^{r-1}} (a\otimes b\otimes c)^{\otimes d}$$

is equal to $\psi(\mathcal{S}_d \otimes \mathcal{S}_d \otimes \mathcal{S}_d)$, the sum of all $\mathcal{T}_{\mathrm{D}}$ where D is a colored Brauer diagram on d vertices

Computing the Norm

$$\|\mathcal{U}\|_{\sigma,4}^4 = \int (\mathcal{U} \cdot a \otimes b \otimes c)^d = \int \mathcal{U}^{\otimes d} \cdot (a \otimes b \otimes c)^{\otimes d} =$$
$$= \mathcal{U}^{\otimes d} \cdot \left(\int (a \otimes b \otimes c)^{\otimes d} \right)$$

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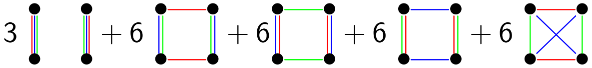
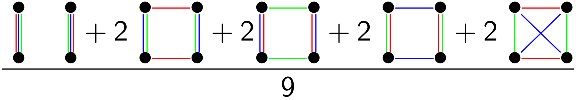
note that $\mathcal{I}_{\mathrm{D}}$ does not depend on the labeling of the vertices, so up to a scalar, $\|\mathcal{U}\|_{\sigma,4}^4$ is equal to



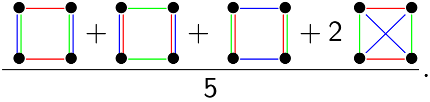
(by abuse of notation, the diagram D represents $\mathcal{I}_{\mathrm{D}}$)

$$\|\mathcal{U}\|_{\sigma,4}^4 =$$



a similar norm that in some ways is a better approximation of the spectral norm is given by

$$\|\mathcal{U}\|_{\#}^4 =$$



Tensor Amplification

if matrix A has singular values $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$
then AA^tA has singular values $\lambda_1^3, \lambda_2^3, \ldots$
so the map $A \mapsto AA^tA$ enhances the low rank structure (largest singular values) and surpresses noise (small singular values)

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we can do something similar with tensors and define

$$\Phi_{\sigma,4} = \nabla \| T \|_{\sigma,4}^4$$

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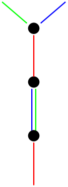
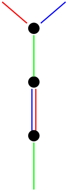
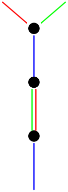
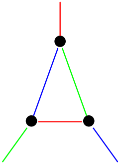
these maps are $\mathrm{O}_p \times \mathrm{O}_q \times \mathrm{O}_r$-equivariant and amplify the low rank structure of a tensor

Partial Colored Brauer Diagrams

in partial colored Brauer diagrams, we have

$\Phi_{\#} = \frac{4}{5}$  $+ \frac{4}{5}$  $+ \frac{4}{5}$  $+ \frac{8}{5}$ 

Numerical Experiments

choose a random tensor $30 \times 30 \times 30$ tensor $\mathcal{T}_n = \mathcal{T} + \mathcal{E}$
$\mathcal{T}$ is random rank 1 with $\|\mathcal{T}\| = 1$
$\mathcal{E}$ is random tensor with $\|\mathcal{E}\| = 10$
signal to noise ratio $-20dB$

experiment: use Alternating Least Squares (ALS) algorithm for best rank 1 approximation of $\mathcal{T}_n$ (to recover $\mathcal{T}$)

use ALS with random initial guess
compare with using ALS with amplified initial guess

Results

Random (10 runs)	Max Fit	Total # Iterations	Total Time
Average	0.7136	77.5080	0.0943
Standard Deviation	0.2715	12.0254	0.0159
Quick Rank 1	Fit	# Iterations	Time
Average	0.7848	2.94	0.0177
Standard Deviation	0.1618	1.2345	0.0025
$\Phi_{\sigma,4}$ **and Quick Rank 1**	Fit	# Iterations	Time
Average	0.8010	2	0.0210
Standard Deviation	0.1256	0	0.0027
$\Phi_{\#}$ **and Quick Rank 1**	Fit	# Iterations	Time
Average	0.8178	2	0.0205
Standard Deviation	0.0515	0	0.0025