On algebraic and geometric properties of spectral convex bodies

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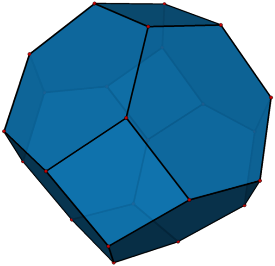

joint work with James Saunderson (Monash University)

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Focus today:

- Operations and metric properties
  Minkowski sums, polarity, volume and Steiner polynomials
- Geometric and algebraic boundary
  faces, algebraic degree, hyperbolicity
- Representations
  spectrahedra and spectrahedral shadows

Spectral convex sets

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Proposition

If K is a symmetric convex set/body, then $\Lambda(K)$ is a convex set/body.

Examples: $\Lambda(K) \;:=\; \{A \in S_2\mathbb{R}^d : \lambda(A) \in K\}$

- Operator norm: $K = \{x : \|x\|_\infty \le 1\} = [-1,1]^d$

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- Schur-Horn orbitopes [S-Sottile-Sturmfels'11]: $K = conv(\mathfrak{S}_d \cdot p)$

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$\delta : \mathbb{R}^d \to S_2\mathbb{R}^d$ diagonal embedding

Lemma

Let K be a symmetric convex set. Then

$$D(\Lambda(K)) \;=\; K \;=\; D(\Lambda(K) \cap \delta(\mathbb{R}^d))\,.$$

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- Again by Lemma: $A = \delta(p) \in \Lambda(K)$. □

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Support functions

For closed convex set $K \subset \mathbb{R}^d$, the support function is $h_K : \mathbb{R}^d \to \mathbb{R}$

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Computation of convex invariants

$K \subset \mathbb{R}^d$ convex body, $B_d = B(\mathbb{R}^d)$ unit ball
Steiner polynomial

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Theorem

$$vol(\Lambda(K) + t \cdot B(S_2\mathbb{R}^d)) \;=\; 2^{\frac{d(d+3)}{2}} \prod_{r=1}^{d} \frac{\pi^{\frac{r}{2}}}{\Gamma(\frac{r}{2})} \int_{K+tB_d} \prod_{i<j} |p_i - p_j| dp$$

If K symmetric polytope, then integral *effectively* computable.

Faces and boundaries

(Exposed) Face of full-dimensional convex body $K \subset \mathbb{R}^d$

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Corollary

$\partial_{\mathrm{alg}} \Lambda(K)$ and $\partial_{\mathrm{alg}} K$ have same degree.

If K hyperbolicity cone, then $\Lambda(K)$ is hyperbolicity cone [Bauschke et al.'01]

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Semidefinite programming

$$\begin{aligned} \max \;\; & \langle c, x \rangle \\ \text{subj. to} \;\; & B(x) \succeq 0 \end{aligned}$$

generalizes linear programming, fast algorithms

Spectrahedra

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where $B_0, \ldots, B_d \in S_2\mathbb{R}^d$ and $\succeq 0$ means positive semidefinite.

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Theorem

If K is a polyhedron, then $\Lambda(K)$ is a spectrahedron.

Schur-Horn orbitopes are spectrahedra

Let $p = (p_1 \geq p_2 \geq \cdots \geq p_d)$
Permutahedron $\Pi(p) \; = \; conv\big\{(p_{\sigma(1)}, \ldots, p_{\sigma(d)}) : \sigma \in \mathfrak{S}_d\big\}$.

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Proposition

$x \in \Pi(p)$ if and only if x majorized by p. That is, $\sum_i x_i = \sum_i p_i$

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Theorem (S-Sottile-Sturmfels'11)

Let $A \in S_2\mathbb{R}^d$. Then $A \in \Lambda(\Pi(p))$ if and only if $\mathrm{tr}(A) = \sum_i p_i$ and

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Schur-Horn orbitope is a spectrahedron of order $2^d - 2$.
For p generic, algebraic degree of $\Pi(p)$ is $2^d - 2$.

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$a \in \mathbb{R}^d$, $b \in \mathbb{R}$ define

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Yes, for $d = 2$: $A \mapsto \mathcal{L}_a(A) = a_1 A + a_2 A^{\mathrm{adj}}$

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$$a^{\mathcal{I}} \ := \ (a_1 - a_2)\mathbf{1}_{I_1} + (a_2 - a_3)\mathbf{1}_{I_1} + \cdots + a_d \mathbf{1}_{I_d},$$

where $\mathbf{1}_{I_j} \in \{0,1\}^d$ are characteristic vectors.

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If $\mathcal{I}$ is a chain, that is, $I_1 \subset I_2 \subset \cdots \subset I_d$, then $a^{\mathcal{I}} = \sigma \cdot a$ for some $\sigma \in \mathfrak{S}_d$.

Proposition

$$P_{a,b} = \{x \in \mathbb{R}^d : \langle a^{\mathcal{I}}, x \rangle \leq b \textit{ for } \mathcal{I} \textit{ numerical chain}\}.$$

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Recall that for $A, B \in S_2\mathbb{R}^d$, $A \otimes B$ is symmetric $d^2 \times d^2$-matrix with eigenvalues $\lambda_i(A) \cdot \lambda_j(B)$ for $1 \leq i, j \leq d$

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Define $\widehat{\mathcal{L}}_a : \bigwedge^1 \mathbb{R}^d \otimes \bigwedge^2 \mathbb{R}^d \otimes \cdots \otimes \bigwedge^d \mathbb{R}^d \to \bigwedge^1 \mathbb{R}^d \otimes \bigwedge^2 \mathbb{R}^d \otimes \cdots \otimes \bigwedge^d \mathbb{R}^d$ by

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Eigenvalues of $\widehat{\mathcal{L}}_a(A)$ are $\langle a^{\mathcal{I}}, \lambda(A)\rangle$ for all $\mathcal{I}$ numerical chains.

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Theorem

If $P = P_{a_1,b_1} \cap \cdots \cap P_{a_M,b_M}$ is a symmetric polyhedron, then

$$\Lambda(P) \;=\; \{A \in S_2\mathbb{R}^d : b_i \cdot \mathrm{Id} - \widehat{\mathcal{L}}_{a_i}(A) \succeq 0 \text{ for } i = 1, \ldots, M\}\,.$$

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Using a result of Ben-Tal and Nemirovski, we can show

Theorem

If $K \subset \mathbb{R}^d$ is a symmetric spectrahedral shadow of order r, then $\Lambda(P)$ is the projection of a spectrahedron of order $r + 2d^2 - 2d - 2$.
Similarly, if K is a symmetric polyhedron with r orbits of facets.

Spectral zonotopes and spectral arrangements?

Line segment $[-z, z]$ for $z \in \mathbb{R}^d \setminus 0$

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The standard permutahedron for $p = (1, 2, \ldots, d)$

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Nice formulas for volumes and Steiner polynomials, ...

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Question

Is there a nice theory of spectral zonotopes and spectral arrangements?

Spectral convex bodies

A spectral convex set is a set of the form

$$\Lambda(K) \ := \ \{A \in S_2\mathbb{R}^d : \lambda(A) \in K\}\,,$$

where K is a symmetric convex set.

Proposition

$\Lambda(K)$ *is a convex set/body.*

- Rich class of convex sets Closed under intersection Minkowski sum, convex hull, and polarity.
- Geometric and algebraic structure intimately related to that of K Support functions, orbits of faces, algebraic boundary, hyperbolicity
- Representations as (projections of) spectrahedra spectrahedra when K polyhedron; spectrahedral shadow when K spectrahedral shadow