Real Smooth Points

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Definitions

- Given $f_1, \ldots, f_s, g_1, \ldots, g_t \in \mathbb{R}[\mathbf{x}]$

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid f_1(\mathbf{x}) = \cdots = f_s(\mathbf{x}) = 0, g_1(\mathbf{x}) > 0, \ldots, g_t(\mathbf{x}) > 0\}.$$

  is an **atomic semi-algebraic set**. If $t = 0$, S is a **real algebraic set**.

- A point $\mathbf{z} \in S$ is **smooth** (or nonsingular) in S if z is smooth in the algebraic set

$$V(f_1, \ldots, f_s) = \{x \in \mathbb{C}^n \ : \ f_1(x) = \cdots = f_s(x) = 0\},$$

  i.e. if there exists a unique irreducible component $V \subset V(f_1, \ldots, f_s)$ containing $\mathbf{z}$ such that

$$\dim T_{\mathbf{z}}(V) = \dim V$$

  where $T_{\mathbf{z}}(V)$ is the tangent space of V at $\mathbf{z}$.

- We denote by $\mathrm{Sing}(S)$ the set of singular (or non-smooth) points in S.

The Problem

Problem

Given S atomic semi-algebraic set.

Find a *smooth* point in each connected component of S

Applications:

- **Kuramoto model:** a dynamical system to model synchronization amongst n coupled oscillators. Computational proof for max number of equilibrium for $n = 4$.
- **Real Dimension:** Try to close the complexity gap between real and complex case. Let $V = V(f_1, \ldots, f_s) \subset \mathbb{C}^n$, $d := \max_i \deg f_i$ and $V_{\mathbb{R}} = V \cap \mathbb{R}^n$.
  Best known algorithms:
  - Find $\dim_{\mathbb{C}} V$: $d^{O(n)}$ worst case running time.
  - Find $r := \dim_{\mathbb{R}} V_{\mathbb{R}}$: $d^{O(r(n-r))}$ worst case running time.

Note: Every atomic semialgebraic set is a projection of a real algebraic set.
$\Rightarrow$ We can always rewrite our semialgebraic set as algebraic by adding variables, e.g.

$$g(x) \leq 0 \leftrightarrow g(x) + \gamma^2 = 0$$
$$g(x) < 0 \leftrightarrow \gamma^2 g(x) + 1 = 0$$

Previous Work

Finding smooth points in each connected component of $V_{\mathbb{R}}$

- **Cell decomposition based on sign conditions**
  - Collins' CAD (1975) • Basu, Pollack, Roy (2006) - Chapter 13
- **Critical points of distance function and polar varieties**

  Only guaranteed to work for smooth varieties

  - Seidenberg (1954) • Bank et al. (1997), (2004), (2009), (2010), (2015)
  - Roullier, Roy, Safey El Din (2000) • Aubry, Rouillier, Safey El Din (2002)
  - Safey El Din and Schost (2003) • Faugere et al. (2008)
  - Mork and Piene (2008) • Hauenstein (2012) • Wu and Reid (2013)
  - Draisma et al. (2016) • Safey El Din and Spaenlehauer (2016)
  - Safey El Din, Yang, Zhi (2018) • Elliott, Giesbrecht and Schost (2020)

Computing Real Dimension

- Collins (1975) • Koiran (1999) • Vorobjov (1999) • Basu, Pollack, Roy (2006)
- Safey El Din and Tsigaridas (2013) • Bannwarth and Safey El Din (2015)

Finding Real Smooth Points: Our Approach

Theorem [Harris, Hauenstein, Sz.]

Let $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n]$ and assume that $V := V(f_1, \ldots, f_s) \subset \mathbb{C}^n$ is equidimensional of dimension $n - s$. Suppose that $g \in \mathbb{R}[x_1, \ldots, x_n]$ satisfies the following conditions:

1. $\mathrm{Sing}(V) \cap \mathbb{R}^n \subset V(g)$;
2. $\dim(V \cap V(g)) < n - s$.

Then the set of points where g restricted to $V \cap \mathbb{R}^n$ attains its extreme values intersects each bounded connected component of $(V \setminus \mathrm{Sing}(V)) \cap \mathbb{R}^n$.

Algorithmically: We find the critical points of g in V by solving

$$L \quad := \quad \left\{ \frac{\partial g}{\partial x_i} + \sum_{t=1}^{s} \lambda_t \frac{\partial f_t}{\partial x_i} \ : \ i = 1, \ldots, n \right\} \cup \{f_1, \ldots, f_s\}.$$

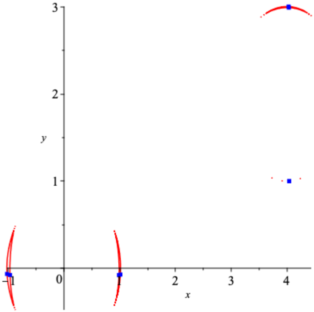
in the variables $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_s$.

Note: The challenge is finding a low degree g satisfying the above two conditions.

I will get back to how to compute such a g at the end of talk.

Example: Mork-Piene Curve

Real plane curve (Mork-Piene 2008): critical points of the distance function from any point in $\mathbb{R}^2$ will not contain smooth points on all four connected components:



$$f_1 = (x^2 + y^2 - 1)((x-4)^2 + (y-2)^2 - 1)$$

$$f_2 = \left(y - \frac{1}{2}\right)\left(y + \frac{1}{2}\right)\left(x - \frac{7}{2}\right)\left(x - \frac{9}{2}\right)$$

$$F = f_1^2 + \frac{1}{100} f_2^3$$

$$\mathbf{g} = (4x^2 - 3)(4y^2 - 1)(4x^2 - 32x + 63)(4y^2 - 16y + 13)$$

Application to Kuramoto Model for $n = 4$

Kuramoto model (1975): a dynamical system to model synchronization amongst n coupled oscillators.
Open problem: Find the maximum number of equilibria for $n \geq 4$.
Polynomial system: Compute max number isolated real solutions of $F = 0$ as $\omega \in \mathbb{R}^3$ for

$$F(s, c; \omega) = \left\{ \omega_i - \tfrac{1}{4} \textstyle\sum_{j=1}^{4} (s_i c_j - s_j c_i), s_i^2 + c_i^2 - 1, s_4, c_4 - 1, \text{ for } i = 1, 2, 3 \right\}.$$

Our approach:

1. Compute the discriminant $D(\omega)$ of the system F: $\deg D(\omega) = 48$.
2. Compute sample points in each bounded connected components of $\mathbb{R}^3 \setminus V(D(\omega))$ computing the critical points of $D(\omega)$, i.e. solve the system:

   $$\nabla D(\omega) = 0 \quad D(\omega) \neq 0.$$

   Note: Bezout bound for this system is $47^3 > 100K$.
3. For each real sample points $\tilde{\omega} \in \mathbb{R}^3$ compute the real solutions of $F(s, c; \tilde{\omega})$.
4. Certify the solutions using `alphaCertified` (Hauenstein, Sottile 2012)

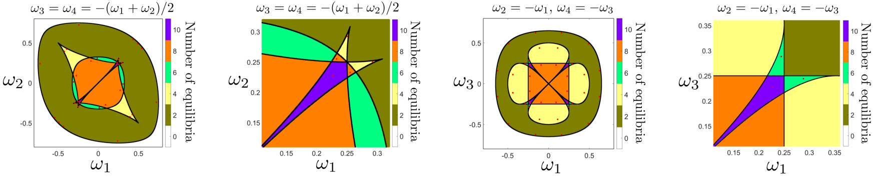
Kuramoto model and symmetries

Add the polynomial $\omega_4 - \frac{1}{4}\sum_{j=1}^{4}(s_4 c_j - s_j c_4)$ to get a system $\bar{F}(s, c; \bar{\omega})$ with $\bar{\omega} \in \mathbb{R}^4$.
Note: $\omega_1 + \omega_2 + \omega_3 + \omega_4 = 0$.
$\rightarrow$ Discriminant $\bar{D}(\bar{\omega})$ of $\bar{F}$ is a symmetric polynomial
$\rightarrow$ $\bar{D}(\bar{\omega}) = H(e)$ with $e = (e_1, \ldots, e_4)$ elementary symmetric polynomials
$\rightarrow$ $\nabla_{\bar{\omega}}\bar{D}(\bar{\omega}) = M \cdot \nabla_e H(e)$ where $\det(M) = \prod_{1 \leq i < j \leq 4}(\omega_i - \omega_j)$

- M non-singular: 105 solutions, orbit size 48 $\rightarrow$ 5040 solutions
- M singular: 4 subsystems, further symmetries: 1292 solutions

Theorem [Harris, Hauenstein, Sz.]

The maximum number of equilibria for the Kuramoto model with $n = 4$ oscillators is 10.



(a) one slice (b) zoomed in (c) other slice (d) zoomed in

Figure 1: Bounded connected regions and critical points, Kuramoto model, $n = 4$.

Computation of Real Dimension

Theorem [Marshall 2008]

For $V \subset \mathbb{C}^n$ an irreducible algebraic set,

$$\dim_{\mathbb{R}} V \cap \mathbb{R}^n = \dim_{\mathbb{C}} V$$

if and only if there exists a smooth $\mathbf{z} \in V \cap \mathbb{R}^n$.

Main idea of a Real Dimension Algorithm:

- If there exists a smooth $\mathbf{z} \in V \cap \mathbb{R}^n$ then $\dim_{\mathbb{R}}(V \cap \mathbb{R}^n) = \dim_{\mathbb{C}}(V)$.
- If not, $V \cap \mathbb{R}^n \subseteq \mathrm{Sing}(V)$ and $\dim_{\mathbb{R}}(V \cap \mathbb{R}^n) < \dim_{\mathbb{C}}(V)$.
- Lower the complex dimension without losing real points, i.e. find $V' \subset V$ algebraic set such that

  $$\dim_{\mathbb{C}} V' = \dim V - 1 \text{ and } V' \cap \mathbb{R}^n = V \cap \mathbb{R}^n.$$

  Next slide: V' is the limit of a perturbed polar varieties.
- Iterate using V' instead of V.

Limits of Perturbed Polar Varieties

$f := \sum_{i=1}^{s} f_i^2 \in \mathbb{R}[x_1, \dots, x_n]$, $\pi_i : \mathbb{C}^n \to C^i$ projection, ε infinitesimal, $V_\varepsilon = V(f - \varepsilon)$.
For $i = 1, \dots n$ the $(i-1)$-th polar variety of V_ε is defined as

$$\mathrm{crit}(V_\varepsilon, \pi_i) := V\left(f - \varepsilon, \frac{\partial f}{\partial x_{i+1}}, \dots, \frac{\partial f}{\partial x_n}\right) \subset \mathbb{C}^n.$$

Theorem [Safey El Din, Tsigaridas 2013]

After a generic change of variables, $\mathrm{crit}(V_\varepsilon, \pi_i)$ are either empty or **smooth and equidimensional of dimension** $i-1$ for $i = 1, \dots, n$.
Furthermore, if $\dim_{\mathbb{R}}(V(f) \cap \mathbb{R}^n) < i$ then

$$V(f) \cap \mathbb{R}^n = \lim_{\varepsilon \to 0} \mathrm{crit}(V_\varepsilon, \pi_i) \cap \mathbb{R}^n.$$

Finding Real Smooth Points on Limit Varieties

Theorem [Harris, Hauenstein, Sz.]

Let $f_1, \ldots, f_s \in \mathbb{R}[\mathbf{x}]$, fix $a = (a_1, \ldots, a_s) \in \mathbb{R}^s$ such that for all sufficiently small $\varepsilon > 0$ $V_\varepsilon := V(f_1 - a_1\varepsilon, \ldots, f_s - a_s\varepsilon) \subset \mathbb{C}^n$ is smooth and equidimensional of dimension $n - s$.
Let $V = \lim_{\varepsilon \to 0} V_\varepsilon$.
Let $g \in \mathbb{R}[\mathbf{x}]$ such that

- $\mathrm{Sing}(V) \cap \mathbb{R}^n \subset V(g)$ and
- $\dim (V \cap V(g)) < n - s$.

Let $C_\varepsilon \subset \mathbb{C}^n$ be the set of critical points of g on V_ε. Then C_ε is finite.
Furthermore, let

$$S := \left(\lim_{\varepsilon \to 0} C_\varepsilon\right) \setminus V(g) \cap \mathbb{R}^n.$$

If $S = \emptyset$, then $V \cap \mathbb{R}^n$ has no bounded connected components of dimension $n - s$. If $S \neq \emptyset$, then $V \cap \mathbb{R}^n$ has some connected components (possibly unbounded) of dimension $n - s$, and S contains smooth points in each of these components.

Note: One can find such g using elimination with degree bound $d^{O(n-s)}$ where $d = \max_i f_i$.

Then we get that the number of critical points in C_ε is at most $d^{O(n(n-s))}$.

Computing g via Isosingular Deflation

For sufficiently small $\varepsilon > 0$ assume $V_\varepsilon := V(f_1 - a_1\varepsilon, \ldots, f_s - a_s\varepsilon) \subset \mathbb{C}^n$ is smooth and equidimensional of dimension $n - s$. Let $\mathbf{z} \in V := \lim_{\varepsilon \to 0} V_\varepsilon \subset \mathbb{C}^n$ generic.

- $F = \{F_1, \ldots, F_N\} \subseteq \mathbb{R}[\mathbf{x}]$ **isosingular deflation** such that:
- $\{f_1, \ldots, f_s\} \subseteq F$
- $F(\mathbf{z}) = 0$
- For $JF(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]^{N \times n}$ Jacobian of F we have $\mathrm{rank}(JF(\mathbf{z})) = s$

Proposition (Simplified version) [Harris, Hauenstein, Sz.]

Assume V is irreducible, $\mathbf{z} \in V$ generic and F is as above. Let $M(\mathbf{x})$ be a generic linear combination of $s \times s$ submatrices of $JF(\mathbf{x})$. For $g(x) := \det(M(\mathbf{x}))$ we have

1. $\mathrm{Sing}(V) \subset V(g)$
2. $\dim(V \cap V(g)) < n - s$.

Furthermore, if the isosingular deflation algorithm takes k iterations then

$$\deg(g) \leq s^{k+1} d$$

Conclusion

Our paper on arXiv: https://arxiv.org/abs/2002.04707.

Future work:

Try to relax the conditions on g using deformations.
Let $V = \lim_{\varepsilon \to 0} V_\varepsilon$ with V_ε smooth and equidimensional of dimension $n - s$ for all sufficiently small $\varepsilon > 0$. Assume that $g \in \mathbb{R}[\mathbf{x}]$ satisfies

1. $\mathrm{Sing}(V) \subset V(g)$
2. $\dim(V_\varepsilon \cap V(g)) < n - s$ for all sufficiently small $\varepsilon > 0$ (replacing $\dim(V \cap V(g)) < n - s$).

What are the limits of the real critical points of g on V_ε in this case?