Real lines on random cubic surfaces

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August 28, 2020

joint work with Rida Ait El Manssour and Mara Belotti

based on [BLLP]:
Saugata Basu, Antonio Lerario, Erik Lundberg, and Chris Peterson. *Random fields and the enumerative geometry of lines on real and complex hypersurfaces.* Math. Ann., 374(3-4):1773–1810, 2019.

Setting

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and consider $f \in \mathbb{K}[x_0, x_1, x_2, x_3]_{(3)}$

Definition

The cubic surface over the field $\mathbb{K}$ associated to f is

$$Z(f) = \{[x_0, x_1, x_2, x_3] \in \mathbb{K}\mathrm{P}^3 \mid f(x_0, x_1, x_2, x_3) = 0\}.$$

$Z(f) \subset \mathbb{K}\mathrm{P}^3$ is smooth for the generic choice of f.

Question:

How many lines are there on a generic cubic surface?

Complex case

Classical algebraic geometry:

Theorem (Cayley, Salmon - 1849)

Every smooth cubic surface over an algebraically closed field contains exactly 27 lines.

Figure: Cubic surface with 27 lines

Real case

No generic answer!

Schläfli (1863)

The number of lines on a real smooth cubic surface is 27, 15, 7 or 3.

The idea is to substitute the word “generic” with the word “random” and ask

Question updated:

What is the expected number of lines on a random cubic surface?

Put a probability distribution on $\mathbb{R}[x_0, x_1, x_2, x_3]_{(3)}$.
Requirements:

- centered
- gaussian
- $O(4)$-invariant

classify such distributions	$\equiv$	classify $O(4)$-invariant scalar products on $\mathbb{R}[x_0, x_1, x_2, x_3]_{(3)}$

Harmonic decomposition

More in general let $W_{n,d} = \mathbb{R}[x_0, \dots x_n]_{(d)}$; $O(n+1)$ acts on $W_{n,d}$

Aim

Find the decomposition of $W_{n,d}$ into its irreducible subrepresentations.

Definition

Define

$$\mathcal{H}_d^n := \{H \in W_{n,d} : \Delta H = 0\}$$

to be the space of real homogeneous harmonic polynomials of degree d in $n+1$ variables.

Decomposition

$$W_{n,d} = \bigoplus_{d-j \in 2\mathbb{N}} \|x\|^{d-j} \mathcal{H}_j^n$$

- Each $\mathcal{H}_j^n$ is $O(n+1)$-invariant and irreducible
- The decomposition is orthogonal w.r.t. every invariant scalar product
- Every invariant scalar product on $\mathcal{H}_j^n$ is a multiple of the L^2 scalar product

Given $f, g \in W_{n,d}$ we can write $f = \sum_j \|x\|^{d-j} f_j$ and $g = \sum_j \|x\|^{d-j} g_j$, with $f_j, g_j \in \mathcal{H}_j^n$, and we have that

$$(f, g) = \sum_{d-j \in 2\mathbb{N}} \mu_j (f_j, g_j)_2$$

for some $\mu_d, \mu_{d-2}, \ldots > 0$.

From scalar products to probability distributions

Fix a harmonic basis $\{H_{j,i}\} \subset \mathcal{H}_j^n$, orthonormal with respect to $(\cdot,\cdot)_2$; then $\{\frac{1}{\sqrt{\mu_j}} H_{j,i}\}$ is an orthonormal basis of $W_{n,d}$ with respect to $(\cdot,\cdot)$.

Random polynomial:

$$P(x) = \sum_{d-j \in 2\mathbb{N}} \lambda_j \sum_{i \in J_j} \xi_{j,i} \|x\|^{d-j} H_{j,i}(x) \qquad \xi_{j,i} \sim N(0,1)$$

where $J_j = dim(\mathcal{H}_j^n)$.

Back to our case

$$W_{3,3} = \mathcal{H}_3^3 \oplus \|x\|^2 \mathcal{H}_1^3$$

Assume that $\lambda_1 + \lambda_3 = 1$, then

$$P_\lambda(x) = \lambda \left(\sum_{j \in J_3} \xi_{3,j} \cdot H_{3,j}(x) \right) + (1 - \lambda) \|x\|^2 \left(\sum_{j \in J_1} \xi_{1,j} \cdot H_{1,j}(x) \right)$$

where $\xi_{i,j} \sim N(0,1)$ for all i, j and independent.

The distributions we are interested in can be parametrized by the single scalar $\lambda \in [0,1]$.

Kostlan distribution

In particular for $\lambda = \frac{1}{3}$ we get the Kostlan polynomial:

$$P(x) = P_{\frac{1}{3}}(x) = \sum_{|\alpha|=3} \xi_\alpha \cdot \sqrt{\frac{3!}{\alpha_0! \cdots \alpha_3!}}\, x_0^{\alpha_0} \cdots x_3^{\alpha_3}$$

Theorem (Basu, Lerario, Lundberg, Peterson)

The expected number E of real lines on a random Kostlan cubic surface in $\mathbb{RP}^3$ is $E = 6\sqrt{2} - 3$.

Strategy

(1) The Grassmannian.

Let $Gr(2,4)$ denote the Grassmannian of 2-planes in $\mathbb{R}^4$, and let $sym^3(\tau^*_{2,4})$ be the 3^{rd} symmetric power of the cotangent of the tautological bundle on $Gr(2,4)$.

Every $f \in \mathbb{R}[x_0, x_1, x_2, x_3]_{(3)}$ defines a section σ_f of the bundle $sym^3(\tau^*_{2,4})$: $\sigma_f(W) = f|_W$.

$$\begin{array}{c} sym^3(\tau^*_{2,4}) \\ \pi \big\downarrow \ \big\uparrow \sigma_f \\ Gr(2,4) \end{array}$$

The problem of finding the expected number of lines in the surface $Z(P) \subseteq \mathbb{R}\mathrm{P}^3$ becomes computing

$$E = \mathbb{E}\#\{W \in Gr(2,4) \mid \sigma_P(W) = 0\}.$$

(2) Trivialization of the bundle and Kac-Rice formula.

Theorem (Kac-Rice formula)

Let $U \subset \mathbb{R}^N$ be an open set and $X : U \to \mathbb{R}^N$ be a random map such that:

- *X is gaussian;*
- *X is almost surely of class C^1;*
- *for every $t \in U$ the random variable $X(t)$ has a nondegenerate distribution;*
- *the probability that X has degenerate zeroes in U is zero;*

Then, denoting by $p_{X(t)}$ the density function of $X(t)$, for every Borel subset $B \subset U$ we have:

$$\mathbb{E}\#(\{X = 0\} \cap B) = \int_B \mathbb{E}\{|det(JX(t))| \mid X(t) = 0\} p_{X(t)}(0) dt$$

where $JX(t)$ denotes the Jacobian matrix of $X(t)$.

Kac-Rice formula

$+$

trivialization of the (oriented) Grassmannian and its vector bundle

$\Downarrow$

$$\mathbb{E}\#\{\tilde{\sigma}_P = 0\} = \int_U \mathbb{E}\{|det(J(W))| \mid \tilde{\sigma}_P(W) = 0\} p(0, W) \cdot w_{Gr^+(2,4)}(W)$$

FACTS:

- $\mathbb{E}\{|det(J(W))| \mid \tilde{\sigma}_P(W) = 0\}p(0, W)$ is a constant that does not depend on W, because P is $O(4)$ invariant
- $J(W)$ and $\tilde{\sigma}_P(W)$ are independent

$$
\begin{aligned}
E &= \mathbb{E}\#\{W \in Gr(2,4) \mid \sigma_{P_\lambda}(W) = 0\} \\
&= \mathbb{E}\{|det(J(W_0))|\} \cdot p(0, W_0) \cdot vol(Gr(2,4))
\end{aligned}
$$

where $W_0 = \{x_2 = 0, x_3 = 0\}$.

(3) $\mathbb{E}\{|det(J(W_0))|\}$.

Up to a constant the matrix $J(W_0)$ is

$$\hat{J} = \begin{bmatrix} a & 0 & d & 0 \\ \sqrt{2}b & a & \sqrt{2}e & d \\ c & \sqrt{2}b & f & \sqrt{2}e \\ 0 & c & 0 & f \end{bmatrix}$$

where $a, b, c, d, e, f \sim N(0,1)$.

$$\begin{aligned} x &= bf - ce \\ y &= af - cd \\ z &= ae - bd \end{aligned} \quad \rightsquigarrow \quad |det(\hat{J})| = |2xz - y^2|$$

Finally

$$\mathbb{E}\{|det(\hat{J})|\} = \frac{1}{4\pi}\int_{\mathbb{R}^3} |2xz - y^2| \frac{e^{-|(x,y,z)|}}{|(x,y,z)|} dxdydz$$

and so

$$E = 6\sqrt{2} - 3$$

All invariant distributions

Theorem (Ait El Manssour, Belotti, M.)

The expected number of real lines on the zero set of the random cubic polynomial P_λ equals:

$$E_\lambda = \frac{9(8\lambda^2 + (1-\lambda)^2)}{2\lambda^2 + (1-\lambda)^2} \left(\frac{2\lambda^2}{8\lambda^2 + (1-\lambda)^2} - \frac{1}{3} + \frac{2}{3}\sqrt{\frac{8\lambda^2 + (1-\lambda)^2}{20\lambda^2 + (1-\lambda)^2}} \right).$$

Strategy

(1) The Grassmannian: as the Kostlan case

(2) Trivialization of the bundle and Kac-Rice formula: as the Kostlan case
so $E_\lambda = \mathbb{E}\{|det(J(W_0))|\}(\lambda) \cdot p(0, W_0)(\lambda) \cdot vol(Gr(2,4))$

(3) $\mathbb{E}\{|det(J(W_0))|\}$: the key is to find the correct harmonic basis!

$$P_\lambda(x) = \lambda \left(\sum_{j \in J_3} \xi_{3,j} \cdot H_{3,j}(x) \right) + (1-\lambda) \left(\sum_{j \in J_1} \xi_{1,j} \cdot \|x\|^2 H_{1,j}(x) \right)$$

$$J(W_0) = \begin{bmatrix} a-b & 0 & d-e & 0 \\ c & a-b & f & d-e \\ a+b & c & d+e & f \\ 0 & a+b & 0 & d+e \end{bmatrix}$$

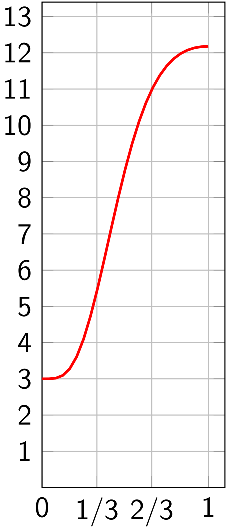
where $a \sim d, b \sim e, c \sim f$ and their variance is a function of λ.

$$\mathbb{E}\{|det(J(W_0))|\} =$$

$$= \frac{1}{4\pi} \int_{\mathbb{R}^3} \left| \left(\frac{\lambda^2}{6} + \frac{(1-\lambda)^2}{48} \right) \lambda^2 x^2 - \frac{\lambda^4}{4} y^2 + \left(\frac{\lambda^2}{6} + \frac{(1-\lambda)^2}{48} \right) \lambda^2 z^2 \right| \cdot$$

$$\cdot \frac{e^{-|(x,y,z)|}}{|(x,y,z)|} \, dxdydz$$

Function E_λ



Maximum: $E_1 = 24\sqrt{\frac{2}{5}} - 3 \simeq 12,179$

The value $\lambda = 1$ corresponds to purely harmonic polynomials of degree 3.

p-adic case (Ait El Manssour)

$f \in \mathbb{Q}_p[x_0, x_1, x_2, x_3]_{(3)}$, $Z(f) \subset \mathbb{Q}_p\mathrm{P}^3$, $\mathrm{GL}_4(\mathbb{Z}_p)$ acts on the space $\mathbb{Q}_p[x_0, x_1, x_2, x_3]_{(3)}$;

$$P(x) = \sum_{|\alpha|=3} \xi_\alpha \, x_0^{\alpha_0} \cdots x_3^{\alpha_3}$$

(1) The Grassmannian

(2) Trivialization of the bundle and Kac-Rice formula

$$\begin{aligned} E &= \lim_{m\to+\infty} \frac{\mu(Gr_{\mathbb{Q}_p}(2,4))\, \lambda(B_m)}{\mu(A_m)} \cdot \frac{\int_{B_m} \mathbb{E}\{|\det(J\psi(t))|_p \mid \psi(t)=0\} \cdot p_{\psi(t)}(0)\, dt}{\lambda(B_m)} \\ &= \frac{\lambda(\mathrm{GL}_4(\mathbb{Z}_p))}{\lambda(\mathrm{GL}_2(\mathbb{Z}_p))\, \lambda(\mathrm{GL}_2(\mathbb{Z}_p))} \cdot \mathbb{E}\left\{ |\det(J\psi(0))|_p \,\middle|\, \psi(0)=0 \right\} \cdot p_{\psi(0)}(0) \\ &= \frac{\lambda(\mathrm{GL}_4(\mathbb{Z}_p))}{\lambda(\mathrm{GL}_2(\mathbb{Z}_p)) \cdot \lambda(\mathrm{GL}_2(\mathbb{Z}_p))} \cdot \mathbb{E}\{|\det(J\psi(0))|_p\} \end{aligned}$$

(3) $\mathbb{E}\{|\det(J\psi(0))|_p\}$

$$M = \begin{bmatrix} a & 0 & d & 0 \\ b & a & e & d \\ c & b & f & e \\ 0 & c & 0 & f \end{bmatrix}$$

and a, b, c, d, e, f are random variables i.i.d uniformly distributed in $\mathbb{Z}_p$.

$$\mathbb{E}\{|\det(M)|_p\} = \lim_{n\to\infty} \frac{1}{p^{6n}} \sum_{0\leq a,b,c,d,e,f\leq p^n-1} |det(M)|_p$$

Theorem (Ait El Manssour)

The expected number of lines on a random cubic surface in $\mathbb{Q}_p\mathrm{P}^3$ is

$$E\#\{\sigma_f = 0\} = \frac{(p^3-1)(p^2+1)}{p^5-1}.$$

Higher dimension

Setting

Every $f \in \mathbb{R}[x_0, \ldots, x_n]_{(d)}$ defines a section σ_f of the bundle $sym^d(\tau^*_{2,n+1})$: $\sigma_f(W) = f|_W$.

$$\begin{array}{c} sym^d(\tau^*_{2,n+1}) \\ \pi \downarrow \;\; \uparrow \sigma_f \\ Gr(2, n+1) \end{array}$$

When $d = 2n - 3$ and f is generic, $\{\sigma_f = 0\}$ is a 0-dimensional submanifold, hence there are finitely many lines on $Z(f)$.

Question:

What is the expected number of lines on a random hypersurface of degree $2n - 3$ in $\mathbb{RP}^n$?

Ideas and intuitions

! Not possible to repeat the strategy of the case $n = 3$!

Possible strategy:

Redo the proof above in more generality, for an implicit harmonic basis

$\rightsquigarrow$ information about the deterministic case

In [BLLP] the authors have proved that

$$\lim_{n\to\infty} \frac{\log E_n^{\mathrm{Kostlan}}}{\log C_n} = \frac{1}{2}$$

Conjecture

$$\lim_{n\to\infty} \frac{\log E_n^{\mathrm{TopHarmonic}}}{\log C_n} > \frac{1}{2}$$