The *p*-adic integral geometry formula

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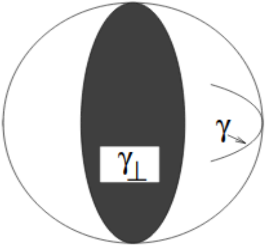

Joint work with Antonio Lerario

Part I: The real world

Integral geometry deals with averaging metric properties under the action of a Lie group.

Applications in: *representation theory, convex geometry, random algebraic geometry, etc.*

The metric

Let $S^n \to \mathbb{P}^n$ be the Hopf fibration. We define

$$d(x, y) = |\sin(\text{angle between the points})|$$

this descends to a metric on $\mathbb{P}^n$. The volume form on the sphere restricts to projective space.

The volume

If Y is a codimension $m-k$ submanifold of a Riemannian manifold M whose volume density is vol and

$$U(Y,\epsilon) = \bigcup_{x \in Y} B(x,\epsilon)$$

denotes the ϵ-neighborhood of Y in M, then

$$\mathrm{vol}_k(Y) := \lim_{\epsilon \to 0} \frac{\mathrm{vol}(U(Y,\epsilon))}{\mathrm{vol}(B_{\mathbb{R}^{m-k}}(0,\epsilon))}.$$

Theorem

Let $X, Y \subseteq \mathbb{P}^n(\mathbb{R})$ be real algebraic sets. Then

$$\int_{\mathrm{SO}_{n+1}(\mathbb{R})} \frac{\mathrm{vol}_z(X \cap gY)}{\mathrm{vol}_z \mathbb{P}^z} dg = \frac{\mathrm{vol}_x X}{\mathrm{vol}_x \mathbb{P}^x} \cdot \frac{\mathrm{vol}_y Y}{\mathrm{vol}_y \mathbb{P}^y}$$

where $x = \dim X, y = \dim Y, z = \dim(X \cap gY)$.

Corollary

The expected number of zeros of a random real polynomial of degree d is $\sqrt{d}$, with respect to a certain distribution.

Theorem

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Corollary

The expected number of zeros of a random real polynomial of degree d *is* $\sqrt{d}$*, with respect to a certain distribution.*

Act II: Leaving the real world.

Definition

The *p*-adic absolute value is defined on $\mathbb{Q}\backslash\{0\}$ by

$$\left|p^e \frac{a}{b}\right|_p := p^{-e} \qquad e \in \mathbb{Z} \text{ and } \gcd(p, ab) = 1, \qquad |0|_p := 0.$$

$$\mathbb{Q}_p = \left\{ \sum_{j \geq e} a_j p^j : e \in \mathbb{Z}, a_e \neq 0, a_j \in \{0, \dots, p-1\} \right\}$$

with the ring structure given by “Laurent series arithmetic”.

- $|a+b|_p \leq \max\{|a|_p, |b|_p\}$, with equality when this maximum is attained exactly once. (*Ultrametric inequality*).
- $|n|_p \leq 1$ for all $n \in \mathbb{Z}$.
- $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ has image $\{0\} \cup \{p^{-n} : n \in \mathbb{Z}\}$.
- $(\mathbb{Q}_p, |\cdot|_p)$ is a metric space.
- For $r \geq 0$, define

$$B(x; r) = \{y \in \mathbb{Q}_p : |y - x|_p \leq r\}.$$

- $\mathbb{Q}_p$ is a locally compact topological space, and ring operations are continuous.

Definition
The *ring of p-adic integers* is defined as

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

The additive group $(\mathbb{Q}_p, +)$ is a locally compact topological group. Thus there is a Haar measure. We normalize $\mu(\mathbb{Z}_p) = 1$.

Lemma

$B(x; r)$ is closed and open in the metric topology. (When $r > 0$)

Lemma

Let $y \in B(x; r)$. Then $B(x; r) = B(y; r)$.

Corollary

$\mathbb{Z}_p$ is the disjoint union of the open and closed balls

$$B(a; p^{-1}), \qquad a \in \{0, 1, \ldots, p - 1\}.$$

Therefore $\mu(B(a; p^{-1})) = p^{-1}$, and $\mu(\mathbb{Z}_p^\times) = 1 - p^{-1}$. ($> 0$!!)

Critical detail: $\mathbb{Q}_p$ *has a totally disconnected topology!!!*

Define the padic unit sphere by

$$S^n_{\mathbb{Q}_p} := \{x \in {\mathbb{Q}_p}^{n+1} : \|x\|_p = 1\}.$$

Remark

The dimension of S^n is $n+1$. Here is a picture:

$(-1, 1)$	$(0, 1)$	$(1, 1)$
$(-1, 0)$	$(0, 0)$	$(1, 0)$
$(-1, -1)$	$(0, -1)$	$(1, -1)$

$$S^n_{\mathbb{Q}_p} := \{x \in {\mathbb{Q}_p}^{n+1} : \|x\|_p = 1\}.$$

We have the *Hopf fibration*

$$\varphi\colon (a_0, \ldots, a_n) \mapsto (a_0 : \ldots : a_n)$$

The spherical metric

$$d(x, y) := \|x \wedge y\|_p$$

gives $\mathbb{P}^n(\mathbb{Q}_p)$ the structure of a metric space.

($\mathbb{R}$: the sine of the angle between $x, y \in S^n_{\mathbb{R}}$ is $\pm\|x \wedge y\|_{\mathbb{R}}$.)

Definition

$\mu(U) := \mu(\mathbb{Z}_p{}^\times)^{-1}\mu(\varphi^{-1}(U))$.

Proposition

A maximal compact subgroup of $\mathrm{GL}_{n+1}(\mathbb{Q}_p)$ *is* $\mathrm{GL}_{n+1}(\mathbb{Z}_p)$*. It is unique up to conjugation.*

The metric on the unit sphere is $\mathrm{GL}_{n+1}(\mathbb{Z}_p)$-invariant.

Definition

Let $X \subseteq \mathbb{P}^n$. For each m define

$$N_m(X) := \frac{\#\{x \pmod{p^m} : x \in \varphi^{-1}(X)\}}{p^m(1-p^{-1})}.$$

The *d-dimensional volume* of $X \subseteq \mathbb{P}^n_{\mathbb{Q}_p}$ is

$$\mathrm{vol}_d(X) := \lim_{m\to\infty} \frac{N_m(X)}{p^{md}}.$$

This definition of volume comes from the theory of zeta functions studied by Denef, Igusa, Oesterlé, Serre, etc.

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Proposition (Serre)

If X is d-dimensional, then the limit in the definition exists and is finite.

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Lemma

This is also the p-adic tube volume. i.e,

$$\mathrm{vol}_a(X) = \lim_{m\to\infty} p^{m(n-a)} \cdot \mu_n\left(\bigcup_{x\in X} B(x, p^{-m})\right).$$

Lemma

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Proof.

Consider the *continuous* map

$$\pi_m \colon X \to \{x \pmod{p^m} : x \in X\}.$$

Choosing an arbitrary set of lifts, we obtain a list of centers needed to cover X. □

Theorem

Let $\mathcal{X}$ be a subscheme of $\mathbb{P}^n$ which is smooth over $\operatorname{Spec}\mathbb{Z}_p$. Then the d-dimensional volume is also the Weil canonical volume of $X = \mathcal{X}(\mathbb{Z}_p)$.

Proof.

By smoothness, the Jacobian is non-vanishing modulo p at every point. We then use a quantitative inverse function theorem to count points modulo p^m.

$$\lim_{m\to\infty} \frac{N_m(X)}{p^{md}} = \frac{\mathcal{X}(\mathbb{F}_p)}{p^d}$$

This turns out to be equal to the Weil canonical volume. □

Examples in $\mathbb{P}^2(\mathbb{Q}_2)$:

$$\mathrm{vol}_1 \mathcal{Z}(x - y + z) = \left(1 + \frac{1}{2}\right), \qquad \mathrm{vol}_1 \mathcal{Z}(x^2 + y^2 - z^2) = 1,$$

$$\mathrm{vol}_1 \mathcal{Z}(2(y^2 - xz)) = \left(1 + \frac{1}{2}\right).$$

Act III: p-adic integral geometry

Theorem (K.-Lerario)

Let $X, Y \subseteq \mathbb{P}^n(\mathbb{Q}_p)$ be algebraic sets. Then

$$\int_{\mathrm{GL}_{n+1}(\mathbb{Z}_p)} \frac{\mathrm{vol}_z(X \cap gY)}{\mathrm{vol}_z \mathbb{P}^z} dg = \frac{\mathrm{vol}_x X}{\mathrm{vol}_x \mathbb{P}^x} \cdot \frac{\mathrm{vol}_y Y}{\mathrm{vol}_y \mathbb{P}^y}$$

where $x = \dim X, y = \dim Y, z = \dim(X \cap gY)$.

Proof sketch

Lemma (Hensel's lemma)

Let $\mathbf{f} = (f_1, \ldots, f_n) \in \mathbb{Z}_p[x_1, \ldots, x_n]^n$*, let* $a \in {\mathbb{Z}_p}^n$*, and let* $J_{\mathbf{f}}(a)$ *be the Jacobian matrix of* $\mathbf{f}$ *at* a*. If*

$$\|\mathbf{f}(a)\| < |J_{\mathbf{f}}(a)|^2 \,,$$

then there is a unique $\alpha \in {\mathbb{Z}_p}^n$ *such that*

$$\mathbf{f}(\alpha) = 0 \text{ and } \alpha \equiv a \pmod{p^m},$$

where $m := 1 - \log_p |J_{\mathbf{f}}(a)|$*.*

Lemma (Linear Approximation Lemma)

Let $X_1, \ldots, X_s \subseteq \mathbb{P}^n$ be algebraic sets such that $\sum_{j=1}^{s} \operatorname{codim}(X_j) = n$. Let $x^{(j)} \in X_j$ be smooth points, and denote by $U_{x^{(j)}}$ balls of $\mathbb{P}^n$ of radius p^{-m} centered at these points. Assume that in these balls we have local equations

$$U_{x^{(j)}} \cap X_j = \{\mathbf{f}_{x^{(j)}} = 0\}.$$

If $\left|J\left(\mathbf{f}_1(x^{(1)}), \ldots, \mathbf{f}_s(x^{(s)})\right)\right|^2 > p^{-m}$, then

$$\#\bigcap_{j=1}^{s}(X_j \cap U_{x^{(j)}}) \;=\; \#\bigcap_{j=1}^{s}(T_{x^{(j)}}X_j \cap U_{x^{(j)}}) \;=\; 0 \text{ or } 1.$$

Proof.

Apply Hensel's lemma in each ball. □

Proposition

For A an open compact subset of an a-dimensional algebraic set in $\mathbb{P}^n_{\mathbb{Q}_p}$*, we have*

$$\int_{\mathrm{GL}_{n+1}(\mathbb{Z}_p)} \frac{\mathrm{vol}_k(A \cap gH)}{\mathrm{vol}_k(\mathbb{P}^k_{\mathbb{Q}_p})}\, dg = \frac{\mathrm{vol}_a(A)}{\mathrm{vol}_a(\mathbb{P}^a_{\mathbb{Q}_p})}.$$

$$\int_{\mathrm{GL}_{n+1}(\mathbb{Z}_p)} \#\left(A \cap gL\right) dg$$

Convert a variety into a union of many tangent spaces (Linear approximation lemma)

$$= \lim_{m\to\infty} \int_{\mathrm{GL}_{n+1}(\mathbb{Z}_p)} \sum_{i=1}^{N_m(A)} \#\left(B(u_i, p^{-m}) \cap T_{u_i}A_i \cap gL\right)$$

Prove the result for pieces of linear varieties, then

$$= \lim_{\ell\to\infty} N_m(A) \cdot \frac{\mathrm{vol}_a(B(u, p^{-m}) \cap \mathbb{P}^a)}{\mathrm{vol}_a(\mathbb{P}^a)}$$

Use the definition of volume

$$= \frac{\mathrm{vol}_a(A)}{\mathrm{vol}_a(\mathbb{P}^a_{\mathbb{Q}_p})}.$$

Applications

Theorem (Oesterlé, weak form)

For an equidimensional variety $A \subseteq \mathbb{P}^n_{\mathbb{Q}_p}$ of dimension d, we have

$$N_m(A) \leq \deg(A)\mathrm{vol}_d(\mathbb{P}^d_{\mathbb{Q}_p}) \cdot p^{md} + o(1).$$

Proof.

$$\int_{\mathsf{GL}_{n+1}(\mathbb{Z}_p)} \frac{\mathrm{vol}_k(A \cap gH)}{\mathrm{vol}_k(\mathbb{P}^k_{\mathbb{Q}_p})}\, dg = \frac{\mathrm{vol}_a(A)}{\mathrm{vol}_a(\mathbb{P}^a_{\mathbb{Q}_p})}.$$

The integrand is at most the degree almost everywhere. □

Corollary

Let

$$g(t) := \zeta_0 + \zeta_1 t + \ldots + \zeta_d t^d.$$

The expected number of zeros of $g(t)$ in $\mathbb{Q}_p$ is 1.

Proof.

Check that the standard Veronese is an isometry. Applying the Integral geometry formula gives the result. □

Theorem (Evans, univariate)

Let

$$g(t) := \zeta_0 + \zeta_1 \binom{t}{1} + \ldots + \zeta_d \binom{t}{d},$$

where $\{\zeta_k\}_{k=0}^{d}$ *is a family of i.i.d. uniform variables in* $\mathbb{Z}_p$*. Then the expected number of zeroes of* g *contained in* $\mathbb{Z}_p$ *is*

$$p^{\lfloor \log_p d \rfloor} \left(1 + p^{-1}\right)^{-1}.$$

The expected number of zeros outside the unit disk is $\frac{|d|_p p^{-1}}{1+p^{-1}}$*.*

Proof.

Let

$$F\colon (t,1) \to \left(1, \binom{t}{1}, \ldots, \binom{t}{d}\right)$$

be the Mahler Veronese map. The Jacobian of F is

$$\left[0, 1, \ldots, \frac{d}{dt}\binom{t}{d}\right].$$

For $t \in \mathbb{Z}_p$ the absolute value of the largest entry is exactly $p^{\lfloor \log_p d \rfloor}$. The Hopf fibration restricted to the image of F is an isometry, so we can apply the integral geometry formula. □

Thanks!

p-adic Integral Geometry, arXiv:1908.04775