Robust control of a risk-sensitive performance measure

Paul Dupuis

Division of Applied Mathematics
Brown University

R. Atar, A. Budhiraja, R. Wu

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- Robust properties of risk-sensitive control (Jacobson, 1973, D, James and Petersen, 2000, Hansen and Sargent, 2001 & 2008)

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- Robust properties of risk-sensitive control (Jacobson, 1973, D, James and Petersen, 2000, Hansen and Sargent, 2001 & 2008)
- Current work (Atar, Budhiraja, D and Wu, see also D, Katsoulakis, Pantazis and Rey-Bellet 2018)

Stochastic uncertain systems with ordinary performance

Elements of the framework

- *Probability models*, on space $\mathcal{S}$, often a path space

$$P = \text{ nominal (computational, design)} \quad \text{vs} \quad Q = \text{ true (impractical)}$$

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  $$E_Q[f] = E_Q[f(X)]$$

  Here f may combine a cost with dynamics that take random variables under Q (or P) into the system state:

  $$f(w) = \int_0^T c(\mathcal{G}[w](t))dt,$$

  $$\mathcal{G} : W \to X, \quad dX(t) = b(X(t))dt + dW(t).$$

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- *A notion of distance* between models, here taken to be relative entropy, aka Kullback-Leibler divergence:

$$R\,(Q\,\|P\,) = \begin{cases} E_Q\left[\log \frac{dQ}{dP}\right] = \int_S \log\left(\frac{dQ}{dP}(s)\right) Q(ds) & \text{if } Q \ll P \\ \infty & \text{else.} \end{cases}$$

Defines neighborhoods of P via $\{Q : R\,(Q\,\|P\,) \leq r\}$.

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- Optimality (tightest bounds with respect to neighborhoods). This automatically introduces nonlinearity, akin to Legendre transform. E.g., if performance measure $E_Q[f]$, Lagrange multipliers lead to quantities like

$$\Lambda_P(\lambda, f) = \sup_Q \left[E_Q[f] - \lambda R(Q\,\|P)\right].$$

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Elements of the framework

- The mapping $f : \mathcal{S} \to \mathbb{R}$ may include parameter $\alpha \in A$ to optimize $f = f_\alpha$. Then may want to solve problems like

$$\min_{\alpha \in A} \max_{Q : R(Q\|P) \le r} E_Q[f_\alpha].$$

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- Key is the *variational formula* relating QoI under Q with functional of P is

$$\log E_P\left[e^{cf}\right] = \sup_{Q \ll P}\left[cE_Q[f] - R\left(Q \| P\right)\right].$$

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$$\log E_P\left[e^{cf}\right] = \sup_{Q \ll P} \left[cE_Q[f] - R(Q\|P)\right].$$

  Hence whenever $Q \ll P$,

$$cE_Q[f] \le R(Q\|P) + \log E_P\left[e^{cf}\right].$$

  Minimizing Q^* is $dQ^* = e^{cf} dP / \int e^{cf} dP$.

Stochastic uncertain systems with ordinary performance

Example: how parts come together

Suppose $f = f_\alpha$ with $\alpha \in A$ and we want to solve "optimally robust optimization": with $r > 0$ fixed

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Then using Lagrange multipliers ($\lambda = 1/c$)

$$\begin{aligned}
&\min_{\alpha \in A} \left[\max_{Q} \min_{c>0} \left(E_Q[f_\alpha] + \frac{1}{c}[r - R(Q \| P)] \right) \right] \\
&= \min_{\alpha \in A} \left[\min_{c>0} \max_{Q} \left(E_Q[f_\alpha] - \frac{1}{c} R(Q \| P) \right) + \frac{1}{c} r \right] \\
&= \min_{\alpha \in A} \min_{c>0} \frac{1}{c} \left(r + \log E_P \left[e^{c f_\alpha} \right] \right).
\end{aligned}$$

Final problem phrased purely in terms of the *design* model, with nice properties in c.

Stochastic uncertain systems with ordinary performance

If for some fixed performance requirement $B < \infty$ we find r such that

$$\min_{\alpha \in A} \min_{c>0} \frac{1}{c}\left(r + \log E_P\left[e^{cf_\alpha}\right]\right) = B.$$

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Then with α^* the minimizer

$$E_Q[f_{\alpha^*}] \leq B$$

for all $Q : R\,(Q \,\|P\,) \leq r$, and r is largest possible value.

Stochastic uncertain systems with ordinary performance

Special case: uncertain model aspects of Jacobson's LEQG. In the 70s Jacobson introduced the linear/exponential/quadratic/Gaussian formulation of control design. Here choose $m(\cdot, \cdot)$ to minimize in

$$S^c(x_0) = \inf_m E\left[\exp c \int_0^T \left(\langle X(s), QX(s)\rangle + \langle u(s), Ru(s)\rangle\right) ds\right]$$

with $u(s) = m(X(s), s)$ and

$$dX(s) = AX(s)ds + Bu(s)ds + CdW(s), \quad X(0) = x_0.$$

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For optimal feedback control m a PDE argument gives

$$\frac{1}{c}\log S^c(x_0)$$
$$= \sup_v E\left[\int_0^T \left(\langle \bar{X}(s), Q\bar{X}(s)\rangle + \langle \bar{u}(s), R\bar{u}(s)\rangle\right) ds - \frac{1}{c}\int_0^T \|v(s)\|^2\, ds\right],$$

where sup over progressively measurable v and

$$d\bar{X}(s) = A\bar{X}(s)ds + B\bar{u}(s)ds + Cv(s)ds + CdW(s), \quad X(0) = x_0.$$

Stochastic uncertain systems with ordinary performance

Thus for any v

$$E \int_0^T \left(\langle \bar{X}(s), Q\bar{X}(s) \rangle + \langle \bar{u}(s), R\bar{u}(s) \rangle \right) ds \\ \leq \frac{1}{c} E \int_0^T \|v(s)\|^2 \, ds + \frac{1}{c} \log S^c(x_0).$$

Can use v to represent model error [e.g., if Ax should be $Ax + Ca(x)$ take $v(s) = a(\bar{X}(s))$].

Deterministic uncertain systems with ordinary performance

H^∞**-control** (state space formulation, adapted to context). A completely deterministic approach uses

$$\dot{\phi}(s) = A\phi(s) + Bu(s) + Cv(s), \quad \phi(0) = x_0,$$

with $u(s) = m(\phi(s), s)$ and $v(s) : [0, T] \to \mathbb{R}^k$ a "disturbance." The control $m(\cdot, \cdot)$ is chosen to minimize in

$$V(x_0) = \inf_{m(\cdot,\cdot)} \sup_{v} \left[\int_0^T \left(\langle \phi(s), Q\phi(s) \rangle + \langle \bar{u}(s), R\bar{u}(s) \rangle - \frac{1}{2c} \|v(s)\|^2 \right) ds \right].$$

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If m is a minimizer, then for any disturbance v

$$\int_0^T \left(\langle \phi(s), Q\phi(s) \rangle + \langle u(s), Ru(s) \rangle \right) ds \leq \int_0^T \frac{1}{2c} \|v(s)\|^2 ds + V(x_0).$$

Here an original motivation was that v could represent model error [e.g., if Ax should be $Ax + Ca(x)$ use $v(s) = a(\phi(s))$].

Stochastic uncertain systems with rare event perfomance

The variational bound based on relative entropy

$$cE_Q[f] \leq R\left(Q \,\|\, P\right) + \log E_P\left[e^{cf}\right]$$

is not useful when $E_Q[f]$ is determined by rare events (e.g., escape probability under the true).

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is not useful when $E_Q[f]$ is determined by rare events (e.g., escape probability under the true). What is a good replacement for

$$\frac{1}{c}\log E_P\left[e^{cf}\right] = \sup_{Q \ll P}\left[E_Q[f] - \frac{1}{c}R\left(Q \,\|\, P\right)\right]?$$

Rare event performance measures and Rényi divergence

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$$\frac{1}{\gamma} \log E_P \left[e^{\gamma f} \right] = \sup_{Q \ll P} \left[\frac{1}{\beta} \log E_Q \left[e^{\beta f} \right] - \frac{1}{\gamma - \beta} R_{\frac{\gamma}{\gamma - \beta}}(Q \,\|\, P) \right],$$

where for mutually absolutely continuous P, Q and $\alpha > 1$

$$R_\alpha(Q \,\|\, P) = \frac{1}{\alpha(\alpha - 1)} \log \int_S \left(\frac{dQ}{dP} \right)^{\alpha - 1} dQ.$$

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Recent variational formula relates risk-sensitive QoI and Rényi divergence. Let $0 < \beta < \gamma$. Then

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As $\beta \downarrow 0$ recover relative entropy formula. Bounds on risk-sensitive QoI for various Q at level β in terms of one at level γ in terms of design:

$$\frac{1}{\beta} \log E_Q \left[e^{\beta f} \right] \leq \frac{1}{\gamma} \log E_P \left[e^{\gamma f} \right] + \frac{1}{\gamma - \beta} R_{\frac{\gamma}{\gamma - \beta}}(Q \,\|\, P).$$

Rare event performance measures and Rényi divergence

Some qualitative properties of Rényi divergence:

- Bounds independent of underlying probability space (data processing inequality)
- A chain rule for product measures, but not for Markov measures
- However, bounds still scale with meaningful limits large time/system size (Rényi rate), even for Markov measures
- Quantity one would optimize over in robust design (here γ) appears also in $R_{\frac{\gamma}{\gamma-\beta}}(Q \,\|\, P)$. Complicates formulation of robust optimization

Rare event performance measures and Rényi divergence

Fix a class of models $\mathcal{Q}$ (e.g., $\{Q : R_1(Q \| P) \le r\}$) and define

$$g(\alpha) = \sup\{R_\alpha(Q \| P) : Q \in \mathcal{Q}\}, \quad \alpha \in (1, \infty).$$

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Theorem

Under integrability conditions on f,

$$\sup_{Q \in \mathcal{Q}} \frac{1}{\beta} \log E_Q e^{\beta f} \leq \inf_{\gamma \geq \beta} F(\beta, \gamma), \quad F(\beta, \gamma) = \left[\frac{g\left(\frac{\gamma}{\gamma - \beta}\right)}{\gamma - \beta} + \frac{1}{\gamma} \log E_P \left[e^{\gamma f} \right] \right],$$

and the infimum over γ is a convex minimization problem.

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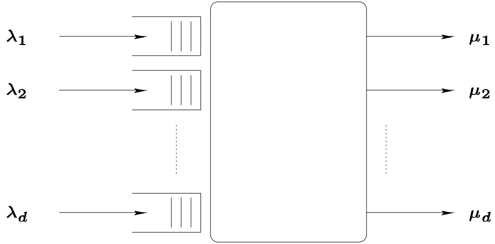
As with ordinary performance measures, there is an optimization/control generalization. Also, the bounds scale properly with time, and one can consider infinite time problem with *Rényi divergence rate*.

Rare event performance measures and Rényi divergence

Example: Optimal optimization/control of tail behavior with model uncertainty

Design model:



Arrivals are Poisson with rates (intensities) λ_i, service (when allocated to i) are exponential with mean $1/\mu_i$, and control is which class to serve. Significant criticism of the model: exponential interarrival and (especially) service times.

Rare event performance measures and Rényi divergence

Let $X_i(t)$ be queue length at time t under some control, $X^n(t) = \frac{1}{n}X(nt)$, and consider as tail-type performance measure

$$E_P e^{\beta \sum_{i=1}^d c_i X_i^n(T)}.$$

[*] *On the risk-sensitive cost for a Markovian multiclass queue with priority*, Atar, Goswami, Shwarz, 2014.

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Then when $n \to \infty$ one can show optimal to allocate service time to solve

$$\min \left\{ \sum_{i=1}^d [\lambda_i e^{\beta c_i} - \rho_i \mu_i (1 - e^{\beta c_i})]^+ : \rho_i \geq 0, \sum_{i=1}^d \rho_i = 1 \right\}.$$

This can be implemented via

$$\text{prioritize service according to largest } \mu_i(1 - e^{-\beta c_i}),$$

a risk-sensitive analogue of μc rule.* But what if not P?

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True model: Let $h_{i,1}$ and $h_{i,2}$ denote hazard rates for times between arrivals and services for class i, and assume

$$a_{i,1} \leq \frac{h_{i,1}(\cdot)}{\lambda_i} \leq b_{i,1}, \quad a_{i,2} \leq \frac{h_{i,2}(\cdot)}{\mu_i} \leq b_{i,2},$$

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and let $\mathcal{Q}$ be corresponding family of models. Then for $Q \in \mathcal{Q}$ we have

$$R_\alpha(Q_{[0,nT]} \| P_{[0,nT]}) \leq nTg_0(\alpha)$$

with the constraint tight for some such Q, and

$$g_0(\alpha) = \sum_{i=1}^{d} \left[k_\alpha(a_{i,1}) \vee k_\alpha(b_{i,1})\right] \lambda_i + \sum_{i=1}^{d} \left[k_\alpha(a_{i,2}) \vee k_\alpha(b_{i,2})\right] \mu_i,$$

with

$$k_\alpha(x) = \frac{x^\alpha - \alpha x + \alpha - 1}{\alpha(\alpha - 1)}.$$

Rare event performance measures and Rényi divergence

For min/max optimum with regard to $\mathcal{Q}$, should solve

$$\min\left\{\frac{g_0\left(\frac{\gamma}{\gamma-\beta}\right)}{\gamma-\beta}+\frac{1}{\gamma}\sum_{i=1}^{d}[\lambda_i e^{\gamma c_i}-\rho_i\mu_i(1-e^{\gamma c_i})]^+ : \rho_i\geq 0, \sum_{i=1}^{d}\rho_i=1\right\},$$

where min is over $\gamma\geq\beta$ and $\{\rho_i\}$.

Summary

- Risk-sensitive control and relative entropy give a useful approach to certain problems of optimization under model uncertainty for ordinary costs.
- Costs based on rare events require a different approach, and we propose a related one based on risk-sensitive control and Renyi divergence.
- Initial applications are to control of queuing models to handle, among other things, old complaints regarding service time distributions.
- Tightness of the bounds, in the sense that there is a model within $\mathcal{Q}$ for which the bounds give equality, has been established for some circumstances (e.g. $\beta > 0$ small), but is an area that needs more investigation.

References

Jacobson's LQEG control problem:

- Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic differential games, D.H. Jacobson, *IEEE Trans. on Auto. Control*, **18**, (1973), 124–131.

Start of H^∞ control:

- Feedback and optimal sensitivity: model reference transformations, multaplicative seminorms, and approximate inverses, G. Zames, *IEEE Trans. Auto. Control*, **26**, (1981), 301–320.

State space formulation of H^∞ control:

- State space formulae for all stabilizing controllers that satisfy an H^∞ norm bound and relations to risk sensitivity, K. Glover and J. C. Doyle, *Systems Control Lett.*, **11**, (1988), 167–172.

Extension to nonlinear systems:

- *Extending H^∞ Control to Nonlinear Systems: Control of Nonlinear Systems to Achieve Performance Objectives,* J. W. Helton and M. R. James, (1999), SIAM.

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Paper that considers diffusions with uncertain drift, and makes connections with H^∞ control:

- Robust properties of risk–sensitive control (D, James and Petersen), *Math. of Control, Signals and Systems*, **13,** (2000), pp. 318–332.

Solution for particular classes of models including uncertain linear/quadratic:

- Minimax optimal control of stochastic uncertain systems with relative entropy constraints (Petersen, James and D), *IEEE Trans. on Auto. Control.,* **45,** (2000), pp. 398–412.

Applications to economics:

- Robust control and model uncertainty (Hansen and Sargent), *The American Economic Review*, **91**, (2001), pp. 60-66.
- *Robustness*, (Hansen and Sargent), Wiley, 2008.

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- Sensitivity analysis for rare events based on Rényi divergence (D, M.A. Katsoulakis, Y. Pantazis and L. Rey-Bellet), to appear in *Ann. of Applied Probab*.
- Robust bounds and optimization of tail properties of queueing models via Rényi divergence, R. Atar, A. Budhiraja, D. R. Wu, *preprint*.