Time consistency and optimal stopping of risk averse multistage stochastic programs

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Mathematical Optimization of Systems Impacted by Rare, High-Impact Random Events

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Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathfrak{F}$ be a filtration $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_T$ (a sequence of sigma fields) with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Stopping time is a random variable $\tau\colon \Omega \to \{0, \dots, T\}$ such that $\{\omega \in \Omega\colon \tau(\omega) = t\} \in \mathcal{F}_t$ for $t = 0, \dots, T$. For a random process $Z_0, ..., Z_T$, adapted to the filtration $\mathfrak{F}$, the optimal stopping time problem can be written as

$$\max_{\tau \in \mathfrak{T}} \mathbb{E}[Z_\tau],$$

where $\mathfrak{T}$ is the set of stopping times.

It is tempting to write distributionally robust/risk averse counterpart as

$$\max_{\tau \in \mathfrak{T}} \inf_{Q \in \mathfrak{M}} \mathbb{E}_Q[Z_\tau],$$

where $\mathfrak{M}$ is a family of probability measures on $(\Omega, \mathcal{F})$.

The expectation operator has the following property

$$\mathbb{E}_Q[\cdot] = \mathbb{E}_{Q|\mathcal{F}_0}\left(\mathbb{E}_{Q|\mathcal{F}_1}\left(\cdots \mathbb{E}_{Q|\mathcal{F}_{T-1}}[\cdot]\right)\right), \tag{1}$$

where $\mathbb{E}_{Q|\mathcal{F}_t}$ denotes the conditional expectation. Note that $\mathbb{E}_{Q|\mathcal{F}_0} = \mathbb{E}_Q$ since $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Then

$$\inf_{Q\in\mathfrak{M}} \mathbb{E}_Q[\cdot] \geq \inf_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\mathcal{F}_0}\left(\inf_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\mathcal{F}_1}\left(\cdots \inf_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\mathcal{F}_{T-1}}[\cdot]\right)\right). \tag{2}$$

There is a technical difficulty here since it is not clear what is minimum (inf) of conditional expectations $\mathbb{E}_{Q|\mathcal{F}_t}$.

Let $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$ and suppose that $\mathfrak{M}$ is a set of probability measures absolutely continuous with respect to the reference probability measure P and such that the densities dQ/dP, $Q \in \mathfrak{M}$, form a bounded convex weakly* closed set $\mathfrak{A} \in \mathcal{Z}^*$ in the dual space $\mathcal{Z}^* = L_q(\Omega, \mathcal{F}, P)$. Consider functional $\varrho : \mathcal{Z} \to \mathbb{R}$ defined as

$$\varrho(Z) := \sup_{Q \in \mathfrak{M}} \mathbb{E}_Q[Z] = \sup_{\zeta \in \mathfrak{A}} \int_\Omega \zeta(\omega) Z(\omega) dP(\omega).$$

Its concave counterpart is $\nu(Z) = -\varrho(-Z)$,

$$\nu(Z) = \inf_{Q \in \mathfrak{M}} \mathbb{E}_Q[Z].$$

Functional $\varrho : \mathcal{Z} \to \mathbb{R}$ has the following properties for $Z, Z' \in \mathcal{Z}$;
(i) $\varrho(Z + Z') \preceq \varrho(Z) + \varrho(Z')$, subadditivity,
(ii) if $Z \preceq Z'$, then $\varrho(Z) \le \varrho(Z')$, monotonicity,
(iii) $\varrho(\lambda Z) = \lambda \varrho(Z),\ \ \lambda \ge 0$, positive homogeneity
(iv) $\varrho(Z + a) = \varrho(Z) + a,\ \ a \in \mathbb{R}$, translation equivariance.

Its concave counterpart is $\nu(Z) = -\varrho(-Z)$ inherits properties (ii)-(iv) and is superadditive. Functional ϱ is convex, and ν is concave.

It is said that a functional $\varrho : \mathcal{Z} \to \mathbb{R}$ is (convex) coherent if it satisfies (i)-(iv) (Artzner et al (1999)). By duality (convex) coherent ϱ can be represented in the form

$$\varrho(Z) = \sup_{\zeta \in \mathfrak{A}} \int_\Omega \zeta(\omega) Z(\omega) dP(\omega),$$

for some set of densities $\mathfrak{A} \subset \mathcal{Z}^*$.

Conditional analogues (assuming that $\mathfrak{A}$, and hence ϱ and ν, are law invariant)

$$\varrho_{|\mathcal{F}_t}(Z) := \operatorname*{ess\,sup}_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\mathcal{F}_t}[Z],\ \nu_{|\mathcal{F}_t}(Z) := \operatorname*{ess\,inf}_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\mathcal{F}_t}[Z].$$

Note that $\varrho_{|\mathcal{F}_t}(Z)$ and $\nu_{|\mathcal{F}_t}$ can be viewed as mappings from $\mathcal{Z}_T = L_p(\Omega, \mathcal{F}_T, P)$ to $\mathcal{Z}_t = L_p(\Omega, \mathcal{F}_t, P)$ and the inequality (2) as

$$\nu(\cdot) \geq \nu_{|\mathcal{F}_0}\left(\nu_{|\mathcal{F}_1}\left(\cdots \nu_{|\mathcal{F}_{T-1}}(\cdot)\right)\right).$$

Similarly

$$\varrho(\cdot) \leq \varrho_{|\mathcal{F}_0}\left(\varrho_{|\mathcal{F}_1}\left(\cdots \varrho_{|\mathcal{F}_{T-1}}(\cdot)\right)\right).$$

Note that for $\tau \in \mathfrak{T}$, Ω is the union of the disjoint sets

$$\Omega_t^\tau := \{\omega\colon \tau(\omega) = t\},\ t = 0, \dots, T,$$

and hence $\mathbf{1}_\Omega = \sum_{t=0}^T \mathbf{1}_{\{\tau=t\}}$. Moreover $\mathbf{1}_{\{\tau=t\}} Z_\tau = \mathbf{1}_{\{\tau=t\}} Z_t$ and thus for $Z_t \in \mathcal{Z}_t$ it follows that

$$Z_\tau = \sum_{t=0}^T \mathbf{1}_{\{\tau=t\}} Z_\tau = \sum_{t=0}^T \mathbf{1}_{\{\tau=t\}} Z_t,$$

and hence (since $\mathbf{1}_{\{\tau=t\}} Z_t$ is $\mathcal{F}_t$-measurable)

$$\begin{aligned} \mathbb{E}(Z_\tau) &= \mathbb{E}\left[\textstyle\sum_{t=0}^T \mathbf{1}_{\{\tau=t\}} Z_t\right] \\ &= \mathbf{1}_{\{\tau=0\}} Z_0 + \mathbb{E}_{|\mathcal{F}_0}\Big(\mathbf{1}_{\{\tau=1\}} Z_1 + \cdots + \mathbb{E}_{|\mathcal{F}_{T-1}}(\mathbf{1}_{\{\tau=T\}} Z_T)\Big). \end{aligned}$$

Definition 1 *Let* $\varrho_{t|\mathcal{F}_t} \colon \mathcal{Z}_{t+1} \to \mathcal{Z}_t$, $t = 0, \dots, T-1$, *be monotone, translation equivariant mappings and consider the corresponding mappings* $\rho_{s,t} \colon \mathcal{Z}_t \to \mathcal{Z}_s$ *represented in the nested form*

$$\rho_{s,t}(\cdot) := \varrho_{s|\mathcal{F}_s}\Big(\varrho_{s+1|\mathcal{F}_{s+1}}\Big(\cdots \varrho_{t-1|\mathcal{F}_{t-1}}(\cdot)\Big)\Big),\ 0 \le s < t \le T.$$

The stopping risk measure *is*

$$\rho_{0,T}(Z_\tau) = \mathbf{1}_{\{\tau=0\}} Z_0 + \varrho_{0|\mathcal{F}_0}\Big(\mathbf{1}_{\{\tau=1\}} Z_1 + \cdots + \varrho_{T-1|\mathcal{F}_{T-1}}(\mathbf{1}_{\{\tau=T\}} Z_T)\Big),$$

and its concave counterpart

$$\nu_{0,T}(Z_\tau) = \mathbf{1}_{\{\tau=0\}} Z_0 + \nu_{0|\mathcal{F}_0}\Big(\mathbf{1}_{\{\tau=1\}} Z_1 + \cdots + \nu_{T-1|\mathcal{F}_{T-1}}(\mathbf{1}_{\{\tau=T\}} Z_T)\Big).$$

Distributionally robust/risk averse optimal stopping

$$\max_{\tau \in \mathfrak{T}} \nu_{0,T}(Z_\tau) \tag{3}$$

or

$$\max_{\tau \in \mathfrak{T}} \rho_{0,T}(Z_\tau).$$

If $\varrho_{t|\mathcal{F}_t}$ are convex coherent, then the composite functional $\rho_{0,T}$ (functional $\nu_{0,T}$) is convex (concave) coherent, and hence

$$\nu_{0,T}(Z) = \inf_{\zeta \in \widehat{\mathfrak{A}}} \int_\Omega \zeta(\omega) Z(\omega) dP(\omega),$$

for some set of densities $\widehat{\mathfrak{A}} \subset \mathcal{Z}^*$. Thus for the corresponding set of probability measures $\widehat{\mathfrak{M}} = \{Q : dQ/dP \in \widehat{\mathfrak{A}}\}$, problem (3) can be written as

$$\max_{\tau \in \mathfrak{T}} \inf_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_Q[Z_\tau].$$

Dynamic programming equations.

Definition 2 (Snell envelope) *Let $Z_t \in \mathcal{Z}_t$, $t = 0, ..., T$, be a stochastic process. The Snell envelope (associated with functional $\rho_{0,T}$) is the stochastic process*

$$E_T := Z_T,$$

$$E_t := Z_t \vee \varrho_{t|\mathcal{F}_t}(E_{t+1}),$$

$t = 0, \dots, T-1$, *defined in backwards recursive way.*

Similarly Snell envelope can be defined for $\nu_{0,T}$.

For $m = 0, \ldots, T$, consider $\mathfrak{T}_m := \{\tau \in \mathfrak{T} \colon \tau \geq m\}$, the optimization problem

$$\max_{\tau \in \mathfrak{T}_m} \rho_{0,T}(Z_\tau), \tag{4}$$

and

$$\tau_m^*(\omega) := \min\{t \colon E_t(\omega) = Z_t(\omega),\ m \leq t \leq T\},\ \omega \in \Omega.$$

Denote by $\mathfrak{v}_m$ the optimal value of the problem (4). Note the recursive property $\rho_{0,T}(Z_\tau) = \rho_{0,m}(\rho_{m,T}(Z_\tau))$, $m = 1, \ldots, T$.

The following assumption was used by several authors, some refer to it as *local property*,

$$\varrho_{t|\mathcal{F}_t}(\mathbf{1}_A \cdot Z) = \mathbf{1}_A \cdot \varrho_{t|\mathcal{F}_t}(Z), \qquad \text{for all } A \in \mathcal{F}_t,\ t = 0, \ldots, T-1.$$

For coherent law invariant mappings $\varrho_{t|\mathcal{F}_t}$ it always holds.

Recall $\mathfrak{T}_m := \{\tau \in \mathfrak{T} \colon \tau \geq m\}$, $\tau_m^*(\omega) := \min\{t \colon E_t(\omega) = Z_t(\omega),\ m \leq t \leq T\}$ and the respective problem (4) $\max_{\tau \in \mathfrak{T}_m} \rho_{0,T}(Z_\tau)$.

Theorem 1 *Let $\varrho_{t|\mathcal{F}_t} \colon \mathcal{Z}_{t+1} \to \mathcal{Z}_t$, $t = 0, \dots, T-1$, be (convex or concave) monotone translation equivariant mappings possessing local property and $\rho_{s,t}$, $0 \leq s < t \leq T$, be the corresponding nested mappings. Then for $Z_t \in \mathcal{Z}_t$, $t = 0, ..., T$, the following holds:*
(i) for $m = 0, \dots, T$,

$$E_m \succeq \rho_{m,T}(Z_\tau),\ \forall \tau \in \mathfrak{T}_m,$$

$$E_m = \rho_{m,T}(Z_{\tau_m^*}),$$

(ii) the stopping time τ_m^* is optimal for the problem (4),
(iii) if $\hat{\tau}_m$ is an optimal stopping time for the problem (4), then $\hat{\tau}_m \succeq \tau_m^*$,
(iv) $\mathfrak{v}_m = \rho_{0,m}(E_m)$, $m = 1, \dots, T$, and $\mathfrak{v}_0 = E_0$.

We have that $E_t \succeq Z_t$, $t = 0, \dots, T$, and

$$\tau_0^*(\omega) = \min\{t\colon Z_t(\omega) \geq E_t(\omega),\ t = 0, \dots, T\}$$

is an optimal solution of the optimal stoping problem, and E_0 is the corresponding optimal value. That is, going forward the optimal stopping time τ_0^* stops at the first time $Z_t = E_t$. As in the risk neutral case the time consistency (Bellman's principle) is ensured here by the decomposable structure of the considered nested risk measure. That is, if it was not optimal to stop within the time set $\{0, \dots, m-1\}$, then starting the observation at time $t = m$ and being based on the information $\mathcal{F}_m$ (i.e., conditional on $\mathcal{F}_m$), the same stopping rule is still optimal for the problem.

For convex law invariant risk functional $\varrho : \mathcal{Z} \to \mathbb{R}$ it holds that $\mathbb{E}[\cdot] \le \varrho(\cdot)$. In that case the distributionally robust formulation will stop later than the corresponding risk neutral formulation. For the respective concave risk functional ν, it will stop earlier.

It is also possible to combine this with policy optimization. That is, to consider problems

$$(\min/\max_{\pi\in\Pi})(\min/\max_{\tau\in\mathfrak{T}})\, \varrho_{0,T}\Big(f_\tau(x_\tau(\cdot),\cdot)\Big),$$

where Π the set of feasible policies $\pi = \{x_0, x_1(\cdot), \dots, x_T(\cdot)\}$ such that $f_t(x_t(\cdot),\cdot) \in \mathcal{Z}_t$, with $f_0 \colon \mathbb{R}^{n_0} \to \mathbb{R}$, $f_t \colon \mathbb{R}^{n_t} \times \Omega \to \mathbb{R}$, and feasibility constraints defined by $\mathcal{X}_0 \subset \mathbb{R}^{n_0}$ and multifunctions $\mathcal{X}_t \colon \mathbb{R}^{n_{t-1}} \times \Omega \rightrightarrows \mathbb{R}^{n_t}$, $t = 1, \dots, T$. It is assumed that $f_t(x_t,\cdot)$ and $\mathcal{X}_t(x_{t-1},\cdot)$ are $\mathcal{F}_t$-measurable.

Some of these formulations preserve convexity of $f_t(\cdot,\omega)$, and some do not.

Interchangeability principle for a functional $\varrho\colon \mathcal{Z} \to \mathbb{R}$, $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$

Consider a function $\psi\colon \mathbb{R}^n \times \Omega \to \mathbb{R} \cup \{+\infty\}$. Let

$$\Psi(\omega) := \inf_{y \in \mathbb{R}^n} \psi(y, \omega)$$

and

$$\mathfrak{Y} := \{\eta\colon \Omega \to \mathbb{R}^n \,|\, \psi_\eta(\cdot) \in \mathcal{Z}\},$$

where $\psi_\eta(\cdot) := \psi\Big(\eta(\cdot), \cdot\Big)$.

Suppose that: the function $\psi(y, \omega)$ is random lower semicontinuous (i.e., its epigraphical mapping is closed valued and measurable), $\Psi \in \mathcal{Z}$ and the functional $\varrho\colon \mathcal{Z} \to \mathbb{R}$ is monotone.

It is said that ϱ is *strictly* monotone if $Z \preceq Z'$ and $Z \neq Z'$ implies that $\varrho(Z) < \varrho(Z')$.

Then:

$$\varrho(\Psi) = \inf_{\eta \in \mathfrak{Y}} \varrho(\psi_\eta) \tag{5}$$

and the implication

$$\bar{\eta}(\cdot) \in \underset{y \in \mathbb{R}^n}{\arg\min}\, \psi(y, \cdot) \implies \bar{\eta} \in \underset{\eta \in \mathfrak{Y}}{\arg\min}\, \varrho(\psi_\eta). \tag{6}$$

holds. If moreover ϱ is strictly monotone, then the converse of (6) holds as well, i.e.,

$$\bar{\eta} \in \underset{\eta \in \mathfrak{Y}}{\arg\min}\, \varrho(\psi_\eta) \implies \bar{\eta}(\cdot) \in \underset{y \in \mathbb{R}^n}{\arg\min}\, \psi(y, \cdot). \tag{7}$$

Since it is assumed that $\psi(y, \omega)$ is random lower semicontinous, it follows that the optimal value function $\Psi(\cdot)$ and the multifunction $\mathfrak{G}(\cdot) := \arg\min_{y \in \mathbb{R}^n} \psi(y, \cdot)$ are measurable. The left hand side of (6) and right hand side of (7) mean that $\bar{\eta}(\cdot)$ is a measurable selection of $\mathfrak{G}(\cdot)$.

As an example consider optimal stopping time of the American put option (this stopping time problem is well-known in mathematical finance)

$$\sup_{\tau \in \mathfrak{T}} \rho_{0,T}\Big(e^{-r\tau}[K - S_\tau]_+\Big),$$

where $\rho_{0,T}$ can be convex or concave stopping risk measure, $K > 0$ is the strike price, $r > 0$ is a fixed discount rate and S_t is the price of the option at time t. It is assumed that S_t follows the geometric random walk process

$$S_t = S_{t-1} \cdot e^{r - \sigma^2/2 + \varepsilon_t},\ t = 1, \ldots, T,$$

in discrete time with ε_t being an i.i.d. Gaussian white noise process, $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$.

The dynamic programming equations (Snell envelope)

$$E_T(S_T) = e^{-rT}[K - S_T]_+,$$

$$E_t(S_t) = e^{-rt}[K - S_t]_+ \vee \varrho_{t|\mathcal{F}_t}\Big(E_{t+1}(S_t \cdot e^{r-\sigma^2/2+\varepsilon_{t+1}})\Big),$$

$t = T-1, \dots, 0$. Here S_t are treated as state variables and ε_t, $t = 0, ..., T$, form a random process. Note that

$$e^{-rt}[K - S_t]_+ \le E_t(S_t).$$

Optimal stopping

$$\tau^* = \min\Big\{t : e^{-rt}[K - S_t]_+ \ge E_t(S_t),\ t = 0, ..., T\Big\},$$

that is it stops first time $e^{-rt}[K - S_t]_+ = E_t(S_t)$.

Note that $E_t(\cdot)$ is convex, if the stopping risk measure is convex.

Alois Pichler, Rui Peng Liu and Alexander Shapiro, Risk averse stochastic programming: time consistency and optimal stopping, https://arxiv.org/abs/1808.10807