Bootcamp Exercises: Real and Convex Algebraic Geometry
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0. (Warm up) Using an SDP solver, for $c = (1, 1, \pm 1)$, find the solution to

$$\max \quad c_1x + c_2y + c_3z \quad \text{such that} \quad \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0.$$

1. (Spectrahedral Sun) Our semester logo is the variety of the polynomial

$$f = 8\left(-x^6 - y^6 - z^6 + x^4y^2 + x^4z^2 + x^2y^4 + x^2z^4 + y^4z^2 + y^2z^4 - 10x^2y^2z^2 \right.$$
$$\left. +3(x^4 + y^4 + z^4) + 2(x^2y^2 + x^2z^2 + y^2z^2) - 3(x^2 + y^2 + z^2) + 1\right).$$

This is also the determinant of the 7×7 matrix $A(x, y, z) = \sum_{i=1}^{7} \ell_i e_i e_i^T + \ell_8 \mathbf{1}\mathbf{1}^T$, where $e_1, \ldots, e_7$ are the seven unit coordinate vectors in $\mathbb{R}^7$, $\mathbf{1}$ is the all ones vector, and $\ell_1, \ldots, \ell_8$ are the eight affine linear polynomials of the form $1 \pm x \pm y \pm z$. This surface bounds a spectrahedron $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : A(x, y, z) \succeq 0\}$.

- For each rank $r = 1, \ldots, 7$, describe the locus of rank $\leq r$ matrices in the linearspace $\{A(x, y, z) : (x, y, z) \in \mathbb{C}^3\}$.
- What are the extreme points of $\mathcal{S}$? What are their ranks?
- Any cost vector $c = (c_1, c_2, c_3) \in \mathbb{R}^3$ gives rise to the semidefinite program

  $$\max \quad c_1x + c_2y + c_3z \quad \text{such that} \quad A(x, y, z) \succeq 0.$$

  Solve this semidefinite program for $c = (4, 5, 6)$. Experimentally, for each rank r, find the probability that for a random cost vector $c \in \mathbb{R}^3$, the optimal solution is a matrix of rank r.
- What is the algebraic boundary of the convex dual of $\mathcal{S}$?
- (*) Can you write $\mathcal{S}$ as a spectrahedron using 6×6 matrices?
- The eight planes $\ell_1 = 0, \ldots, \ell_8 = 0$ bound a regular octahedron. What are the analogues of this surface for the other platonic solids?

2. (Algebraic boundaries) Let $f = x^3 - 3x + y^2 + z^2$ and $g = 1 - (x - 1)^2 - y^2$.

- For $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : f \geq 0, g \geq 0\}$, find the algebraic boundary of the projection of $\mathcal{S}$ onto the plane $\{x + y + z = 0\}$.
- Describe the convex body dual to $\mathcal{S}$.
- What is the algebraic boundary of the convex hull of the curve defined by $f = 0$ and $g = 0$? And its dual?

3. (Random Spectrahedra)

- Find $d = d_1(n)$ so that a generic d-dimensional affine space in the space of $n \times n$ complex symmetric matrices has finitely many matrices of rank one. How many will there be?
- For $n = 3, 4$, how many can be real? What if the linearspace must contain a positive definite matrix? Try to experiment!
- What about rank two?

4. (Rigidity connections) Given five points $p_1 = (x_1, y_1), \ldots, p_5 = (x_5, y_5) \in \mathbb{R}^2$, for each $1 \le i \le j \le 5$, define

$$d_{ij} = ||p_i - p_j||_2^2 = (x_i - x_j)^2 + (y_i - y_j)^2,$$

and consider the 4×4 matrix $A = A(d)$ with (i, j)th entry equal to $A_{ij} = d_{i5} + d_{j5} - d_{ij}$.

- Check that $A_{ij} = \langle p_i - p_5, p_j - p_5 \rangle$ and use this to show that A is a positive semidefinite matrix of rank ≤ 2.
- What is the dimension of the locus of rank-2 matrices $A(d)$ with

  $$d_{12} = d_{13} = d_{24} = d_{34} = d_{35} = d_{45} = 1?$$

  For $p_1 = (0, 0)$ and $p_2 = (0, 1)$, find points p_3, p_4, p_5 that maximize $d_{15} + 2d_{25}$.
- (*) An easier version of Alt's problem is to fix two base points, (for example, $p_1 = (0, 0)$, $p_2 = (0, 1)$), and ask for a four-bar linkage that whose apex p_5 passes through *five* prescribed points in the plane. How does this condition translate to conditions on distance coordinates d_{ij}? Can semidefinite programming help to find a solution to this problem?