Problem Session for Numerical Algebraic Geometry

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1. Illustrative examples

(1) Numerically approximate the solutions of $f(x) = x^5 - x + 1 = 0$.

(2) The eigenvalue-eigenvector problem for a matrix $A \in \mathbb{C}^{n\times n}$ is naturally $2-$homogeneous defined by
$$f(\lambda, v) = Av - \lambda v = 0 \text{ where } (\lambda, v) \in \mathbb{C} \times \mathbb{P}^{n-1}.$$
- Show that the $2-$homogeneous Bézout bound for f is n.
- Utilize a $2-$homogeneous start system to solve the eigenvalue-eigenvector problem for the following matrices:
$$A_1 = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix},\ A_2 = \begin{bmatrix} -2 & 4 \\ -4 & 6 \end{bmatrix},\ A_3 = \begin{bmatrix} 6 & 2 & -4 & 2 \\ -6 & -2 & 8 & -6 \\ -2 & -1 & 4 & -1 \\ -2 & 0 & 0 & 4 \end{bmatrix}.$$
Interpret the solutions in terms of algebraic and geometric multiplicities.

(3) Setup a parameter homotopy for
$$f(x;p) = \begin{bmatrix} x_1^2 - (p_1+p_2)x_1 + p_1p_2 \\ (x_1 - p_1)x_2 + p_3x_1 + p_4 \end{bmatrix}.$$
Use the parameter homotopy to solve $f(x;p) = 0$ when $p = (-4, 2, 2, -3)$.

(4) Compute a numerical irreducible decomposition for
$$f(x,y,z) = \begin{bmatrix} x(x^2 - y - z^2) \\ x(x+y+z^2-2)(y-5) \\ x(x^2+x-2)(z-2) \end{bmatrix} = 0.$$

(5) For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, compute a numerical irreducible decomposition of
$$A^T A - I = 0.$$
For each irreducible component $V \subset \mathbb{C}^4$, consider $X_t = V \cap \mathcal{H}_t$ where $\mathcal{H}_t$ is defined by
$$a_{11} + 2a_{12} + 3a_{21} + 5a_{22} = t.$$
Compute the linear trace of V, i.e., compute $\alpha, \beta \in \mathbb{C}^4$ such that
$$\alpha \cdot t + \beta = \sum_{x_t \in X_t} x_t.$$

(6) Consider the parameterized polynomial
$$f(x;p) = x^2 - x - p = 0.$$
Verify that $\{-1/4\}$ is the branch locus so we aim to create loops in $\mathbb{C} \setminus \{-1/4\}$. At $p = 0$, we have two solutions, say $x_1 = 0$ and $x_2 = 1$. Perform a monodromy loop that encircles the point $p = -1/4$, e.g., $p(\theta) = -1/4 + 1/4 \cdot e^{i\theta}$, and show that this loop generates a transposition of the roots. Hence, the monondromy group is the symmetric group S_2.

2. Power flow/Kuramoto

For n oscillators, fix $s_n = 0$ and $c_n = 1$, parameters $\alpha \in \mathbb{C}^{n-1}$ and $B \in \mathbb{C}^{n\times n}$ with $B = B^T$, and consider the polynomial system

$$F(s,c;\alpha,B) = \begin{bmatrix} \alpha_i - \sum_{j=1}^n B_{ij}(s_i c_j - s_j c_i) & i = 1,\ldots,n-1 \\ s_i^2 + c_i^2 - 1 & i = 1,\ldots,n-1 \end{bmatrix} = 0$$

which consists of $2(n-1)$ equations in $2(n-1)$ variables.

a. Compute the generic root count for $\alpha \in \mathbb{C}^{n-1}$ and $B = B^T \in \mathbb{C}^{n\times n}$ when $n = 3$ and $n = 4$.

b. Compute the generic root count for $\alpha \in \mathbb{C}^{n-1}$ and $v \in \mathbb{C}^n$ where $B = vv^T$ (i.e., rank 1 coupling case) when $n = 3$ and $n = 4$.

c. For $n = 4$, show that $\alpha = (0.5, 0.5, -0.5, -0.5)$ and $v = (1,1,1,1)$ with $B = vv^T$ has 10 real solutions. What happened to the other solutions? What happens when one slightly perturbs α?

d. For $n = 4$, show that $\alpha = 0$ and $B = \begin{bmatrix} 0 & -3.9524 & 0.3177 & 4.3192 \\ -3.9524 & 0 & 6.3855 & -7.9773 \\ 0.3177 & 6.3855 & 0 & -7.4044 \\ 4.3192 & -7.9773 & -7.4044 & 0 \end{bmatrix}$ (data adapted from Zachary Charles) has 18 real solutions.

e. Experiment with the parameters to show that all solutions can be real in both the arbitrary and rank 1 coupling cases when $n = 3$.

(Open) f. Is 18 the maximum number of real solutions for $n = 4$ with arbitrary coupling ($B = B^T$)?

(Open) g. Is 10 the maximum number of real solutions for $n = 4$ with rank 1 coupling ($B = vv^T$)?

(Open) h. Determine the generic number of solutions as a function of $r = \operatorname{rank} B$ and n. Determine the maximum number of real solutions as a function of $r = \operatorname{rank} B$ and n.

3. PLANE CONICS

a. Show that the space of plane conics in $\mathbb{C}^3$ is 8 dimensional and a general plane conic can be defined by

$$\begin{bmatrix} x^2 + a_1xy + a_2y^2 + a_3x + a_4y + a_5 \\ z + b_1x + b_2y + b_3 \end{bmatrix} = 0$$

for some $(a, b) \in \mathbb{C}^5 \times \mathbb{C}^3$.

b. A classical enumerative geometry problem is to count the number of plane conics in $\mathbb{C}^3$ that pass through k points and intersect $8 - 2k$ lines in general position. The following table lists the degrees based on k:

k	number of plane conics
3	1
2	4
1	18
0	92

Verify the generic root count is 18 for $k = 1$ (without loss of generality, one may take the point to be the origin).

c. Setup a parameter homotopy for $k = 1$ and experiment to find the possible number of real solutions.

(Open) d. Taking the point to be the origin, is it possible to find 6 real lines for which there are 18 nonreal plane conics passing through the origin and intersects the lines? If this is impossible, what structure in the system requires there to always be a real solution when the parameters (which define the real lines) are real?

e. Verify that all 92 plane conics that intersect the lines

$$\mathcal{L}_i = \{p_i + tv_i \mid t \in \mathbb{C}\}$$

are real:

$$\begin{array}{ll}
p_1 = (0.46978, -3.988, -2.3527) & v_1 = (2.9137, 1.546, -0.27448) \\
p_2 = (3.19, 0.5752, 3.0953) & v_2 = (0.56569, 1.108, 4.3629) \\
p_3 = (0.40308, 0.78659, 0.9053) & v_3 = (-3.0656, -1.4638, 1.4096) \\
p_4 = (-4.3743, 4.0046, -1.0243) & v_4 = (-0.9163, 3.6495, -2.6528) \\
p_5 = (1.5198, -0.86125, -4.5963) & v_5 = (-3.8418, 3.9541, 2.5494) \\
p_6 = (0.46801, -4.0308, -2.4411) & v_6 = (1.0225, 1.6422, 1.5925) \\
p_7 = (-3.3382, 3.8432, 1.693) & v_7 = (-4.4657, 1.9618, 1.6865) \\
p_8 = (1.3536, 3.6311, 0.42864) & v_8 = (-3.1442, -2.4915, -0.63586).
\end{array}$$

4. MATRIX VARIETIES

Consider the Zariski closure of $SO(n)$, namely

$$\mathcal{SO}_n = \{A \in \mathbb{C}^{n\times n} \mid A^T A = I,\ \det(A) = 1\},$$

and $SE(n)$, namely

$$\mathcal{SE}_n = \{(A,x,y,r) \in \mathbb{C}^{n\times n} \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \mid A^T A = I,\ \det(A) = 1,\ y + Ax = 0,\ 2r + x^T x = 0\}.$$

a. Compute $\deg \mathcal{SO}_3$ which is the generic number of assembly configurations for spherical pentads.

b. Compute $\deg \mathcal{SE}_2$ which is the generic number of assembly configurations for planar pentads.

c. Compute $\deg SE_3$ which is the generic number of assembly configurations for Stewart-Gough platforms.

d. Verify that all witness points for $\mathcal{SE}_3$ with respect to the linear system

$$\ell_i = r + b_i^T x + p_i^T y + p_i^T M b_i - (b_i^T b_i + p_i^T p_i - d_i^2)/2 = 0, \quad i = 1, \ldots, 6,$$

are real for the following data from Dietmaier (1998):

$$B = \begin{bmatrix} 0 & 1.107915 & 0.549094 & 0.735077 & 0.514188 & 0.590473 \\ 0 & 0 & 0.756063 & -0.223935 & -0.526063 & 0.094733 \\ 0 & 0 & 0 & 0.525991 & -0.368418 & -0.205018 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0.542805 & 0.956919 & 0.665885 & 0.478359 & -0.137087 \\ 0 & 0 & -0.528915 & -0.353482 & 1.158742 & -0.235121 \\ 0 & 0 & 0 & 1.402538 & 0.107672 & 0.353913 \end{bmatrix}$$

$$d = \begin{bmatrix} 1 & 0.645275 & 1.086284 & 1.503439 & 1.281933 & 0.771071 \end{bmatrix}$$

where b_i and p_i is the i^{th} column of B and P, respectively. (This computation verifies that every assembly configuration for a Stewart-Gough platform can be real.)

(Open) e. Determine the maximum number of real witness points for $\mathcal{SO}_N$ and $\mathcal{SE}_N$. Can they all be real?