Graphical models exercises

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Problem 0.1. Let $G = (V, E)$ be an undirected graph and suppose that A, B, and C are disjoint subsets of V such that C does **not** separate A and B. Construct a probability distribution satisfying all the global Markov statements of G and not satisfying $X_A \perp\!\!\!\perp X_B | X_C$. *Hint:* Try constructing a Gaussian distribution.

Problem 0.2. Prove the following statements regarding marginals and conditionals of Gaussian distributions (if you haven't done so in the past).

(a). The marginal of a Gaussian distribution $X \sim \mathcal{N}(\mu, \Sigma)$ is a Gaussian distribution:

$$X_A \sim \mathcal{N}(\mu_A, \Sigma_{A,A}).$$

(b). The conditional of a Gaussian distribution $X \sim \mathcal{N}(\mu, \Sigma)$ is a Gaussian distribution:

$$(X_A | X_B = x_B) \sim \mathcal{N}(\mu_A + \Sigma_{A,B}(\Sigma_{B,B})^{-1}(x_B - \mu_B), \Sigma_{A,A} - \Sigma_{A,B}(\Sigma_{B,B})^{-1}\Sigma_{B,A}).$$

(c). Independence in a Gaussian distribution $X \sim \mathcal{N}(\mu, \Sigma)$ is equivalent to determinants vanishing:

$$X_a \perp\!\!\!\perp X_b \iff \Sigma_{a,b} = 0.$$

(d). Conditional independence in a Gaussian distribution $X \sim \mathcal{N}(\mu, \Sigma)$ is equivalent to a rank condition:

$$X_A \perp\!\!\!\perp X_B | X_C \iff \operatorname{rank}(\Sigma_{A \cup C, B \cup C}) \leq |C|.$$

Problem 0.3. Given a set $\mathcal{C}$ of conditional independence statements for a Gaussian random vector $X \sim \mathcal{N}(\mu, \Sigma)$, we can build the conditional independence ideal $I_{\mathcal{C}}$ containing all equations corresponding to these statements. Often times finding the primary decomposition of $I_{\mathcal{C}}$ gives additional conditional independence statements that X satisfies.

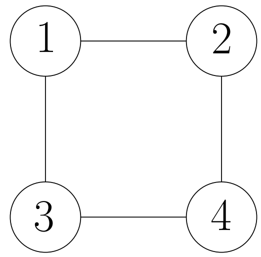
For the following problems it might be easier to use a computer algebra system like Macaulay2. Let $X \sim \mathcal{N}(\mu, \Sigma)$ be a 3-dimensional Gaussian random vector.

(a). Show that the statements $X_1 \perp\!\!\!\perp X_2 | X_3, X_2 \perp\!\!\!\perp X_3$ imply that $X_2 \perp\!\!\!\perp (X_1, X_3)$.

(b). Show that $X_1 \perp\!\!\!\perp X_3 | X_2, X_2 \perp\!\!\!\perp X_3 | X_1$ implies $(X_1, X_2) \perp\!\!\!\perp X_3$.

(c). Show that $X_1 \perp\!\!\!\perp X_3 | X_2, X_1 \perp\!\!\!\perp X_3$ implies that either $X_1 \perp\!\!\!\perp (X_2, X_3)$ or $(X_1, X_2) \perp\!\!\!\perp X_3$.

Problem 0.4. Consider the graph

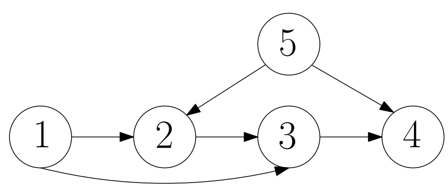


(a). Compute the ideal of the parametrization I_G and the global Markov ideal $I_{\mathrm{global}(G)}$ if the random variable $X \in [2] \times [2] \times [2] \times [2]$ has binary coordinates.

(b). Compute the ideal of the parametrization I_G and the global Markov ideal $I_{\mathrm{global}(G)}$ if the random variable $X \sim \mathcal{N}(0, \Sigma)$ is Gaussian.

The Macaulay2 package "GraphicalModels" might be useful.

Problem 0.5. Consider the graph



(a). Compute the global Markov statements for this DAG.

(b). Compute the ideal of the parametrization I_G and the global Markov ideal $I_{\mathrm{global}(G)}$ if the random variable $X \in [2] \times [2] \times [2] \times [2] \times [2]$ has binary coordinates.

(c). Compute the ideal of the parametrization I_G and the global Markov ideal $I_{\mathrm{global}(G)}$ if the random variable $X \sim \mathcal{N}(0, \Sigma)$ is Gaussian.

The Macaulay2 package "GraphicalModels" might be useful.

Problem 0.6. Let $X \sim \mathcal{N}(\mu, \Sigma)$ be a Gaussian random vector, and let $G = (V, E)$ be a DAG.

(a). Assume that X satisfies the directed global Markov property with respect to G.

1. Show that X satisfies the *directed local Markov property* with respect to to G, i.e. for every $v \in V$,
$$X_v \perp\!\!\!\perp X_{\mathrm{nd}(v) \setminus \mathrm{pa}(v)} | X_{\mathrm{pa}(v)}.$$
Here $\mathrm{nd}(v)$ is the set of non-descendants of v, i.e. all vertices to which there isn't a directed path from v, and $\mathrm{pa}(v)$ is the set of parents of v, i.e. all vertices u such that there is an edge $u \to v$.

2. Now, define the *residuals*

$$\epsilon_i := X_i - \Sigma_{i,\mathrm{pa}(i)}(\Sigma_{\mathrm{pa}(i),\mathrm{pa}(i)})^{-1}X_{\mathrm{pa}(i)}.$$

Show that they are Gaussian random variables and are pairwise independent.

(b). Show that if there exist $\lambda_{ij} \in \mathbb{R}$ for all edges $(i,j) \in E$ and independent Gaussian random variables $\epsilon_1, \dots, \epsilon_n$ such that

$$X_i = \sum_{j\in \mathrm{pa}(i)} \lambda_{ij}X_j + \epsilon_j,$$

then X satisfies the directed local Markov property with respect to G.

Problem 0.7. Classify the Markov equivalence classes of DAGs on 4 vertices.

Problem 0.8. Let $G = (V, D, B)$ be an acylic mixed graph and let X be a Gaussian random vector with covariance matrix

$$\Sigma = (I - \Lambda)^{-T}\Omega(I - \Lambda)^{-1},$$

where $\Lambda \in \mathbb{R}^D$, $\Omega \in PD(B)$.

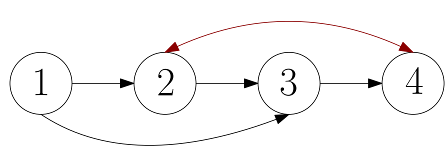
(a). For a directed path $\pi = u_0 \to u_1 \to \cdots \to u_k$, the *path monomial* m_π is defined as

$$m_\pi = \lambda_{u_0u_1}\lambda_{u_1u_2}\cdots\lambda_{u_{k-1}u_k}.$$

Show that the i,j-th entry of $(I - \Lambda)^{-1}$ equals

$$\left((I-\Lambda)^{-1}\right)_{i,j} = \sum_{\text{directed paths } \pi \text{ from } i \text{ to } j} m_\pi.$$

(b). For the following graph



compute $(I - \Lambda)^{-1}$ using part (a).

(c). A **trek** between two vertices i and j in a mixed graph G has the form

1. $i = u_k \leftarrow u_{k-1} \leftarrow \cdots \leftarrow u_0 \to \cdots \to v_{\ell-1} \to v_\ell = j$, or
2. $i = u_k \leftarrow u_{k-1} \leftarrow \cdots \leftarrow u_0 \leftrightarrow v_0 \to \cdots \to v_{\ell-1} \to v_\ell = j$

In both cases k, ℓ are nonnegative integers. For a trek τ the *trek monomial* m_τ is:

$$m_\tau = \lambda_{u_{k-1}u_k} \cdots \lambda_{u_0u_1}\omega_{u_0u_0}\lambda_{u_0v_1} \cdots \lambda_{v_{\ell-1}v_\ell}$$

if the trek is of type 1, and

$$m_\tau = \lambda_{u_{k-1}u_k} \cdots \lambda_{u_0u_1}\omega_{u_0v_0}\lambda_{v_0v_1} \cdots \lambda_{v_{\ell-1}v_\ell}$$

if the trek is of type 2.

Show that the i, j-th entry of the covariance matrix $\Sigma = (I - \Lambda)^{-T}\Omega(I - \Lambda)^{-1}$ equals

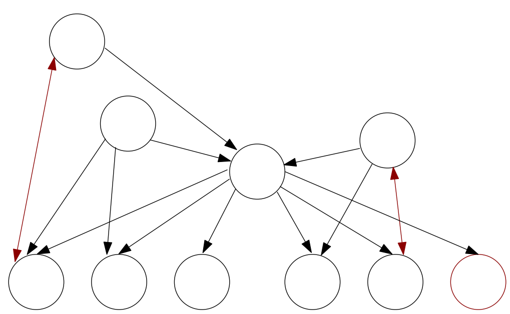
$$\Sigma_{i,j} = \sum_{\text{treks } \tau \text{ between } i \text{ and } j} m_\tau.$$

(d). For the graph from part (b). compute Σ in terms of the entries Λ and Ω using (c).

Trek separation. Let $G = (V, D, B)$ be a mixed graph. Let $A, B, C_A, C_B \subseteq V$. We say that (C_A, C_B) *trek separates* A and B if every trek τ between a vertex in A and a vertex in B either goes through a vertex in C_A on its left side or through a vertex in C_B on its right side.

Theorem 0.9 ([3])**.** The submatrix $\Sigma_{A,B}$ has rank at most r for all $\Sigma \in \mathcal{M}_G$ if and only if there exist C_A, C_B such that (C_A, C_B) trek separates A and B, and $|C_A| + |C_B| \leq r$.

Problem 0.10. For the following graph



compute I_G and $I_{\text{global}(G)}$ using the Macaulay2 package "GraphicalModels". Further, compute the trek separation statements and identify the generators of I_G corresponding to them.

Open Problems. A very good source of open problems regarding linear structural equation problems is Section 3 of [1].

References

[1] M. Drton. *Algebraic Problems in Structural Equation Modeling.* 2016

[2] S. Sullivant. *Algebraic Statistics.* 2018

[3] S. Sullivant, K. Talaska, and J. Draisma. *Trek Separation for Gaussian Graphical Models.* Annals of Statistics 2010