Graphical models exercises

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**Problem 0.1.** Let $G = (V, E)$ be an undirected graph and suppose that $A, B,$ and $C$ are disjoint subsets of $V$ such that $C$ does not separate $A$ and $B$. Construct a probability distribution satisfying all the global Markov statements of $G$ and not satisfying $X_A \perp \!\!\!\!\!\!\! \perp X_B | X_C$. 

*Hint:* Try constructing a Gaussian distribution.

**Problem 0.2.** Prove the following statements regarding marginals and conditionals of Gaussian distributions (if you haven’t done so in the past).

(a). The marginal of a Gaussian distribution $X \sim \mathcal{N}(\mu, \Sigma)$ is a Gaussian distribution:

$$X_A \sim \mathcal{N}(\mu_A, \Sigma_{A,A}).$$

(b). The conditional of a Gaussian distribution $X \sim \mathcal{N}(\mu, \Sigma)$ is a Gaussian distribution:

$$(X_A|X_B = x_B) \sim \mathcal{N}(\mu_A + \Sigma_{A,B}(\Sigma_{B,B})^{-1}(x_B - \mu_B), \Sigma_{A,A} - \Sigma_{A,B}(\Sigma_{B,B})^{-1}\Sigma_{B,A}).$$

(c). Independence in a Gaussian distribution $X \sim \mathcal{N}(\mu, \Sigma)$ is equivalent to determinants vanishing:

$$X_a \perp \!\!\!\!\!\!\! X_b \iff \Sigma_{a,b} = 0.$$  

(d). Conditional independence in a Gaussian distribution $X \sim \mathcal{N}(\mu, \Sigma)$ is equivalent to a rank condition:

$$X_A \perp \!\!\!\!\!\!\! X_B | X_C \iff \text{rank}(\Sigma_{A\cup C,B\cup C}) \leq |C|.$$  

**Problem 0.3.** Given a set $\mathcal{C}$ of conditional independence statements for a Gaussian random vector $X \sim \mathcal{N}(\mu, \Sigma)$, we can build the conditional independence ideal $I_\mathcal{C}$ containing all equations corresponding to these statements. Often times finding the primary decomposition of $I_\mathcal{C}$ gives additional conditional independence statements that $X$ satisfies.

For the following problems it might be easier to use a computer algebra system like Macaulay2. Let $X \sim \mathcal{N}(\mu, \Sigma)$ be a 3-dimensional Gaussian random vector.

(a). Show that the statements $X_1 \perp X_2 | X_3, X_2 \perp X_3$ imply that $X_2 \perp (X_1, X_3)$.

(b). Show that $X_1 \perp X_3 | X_2, X_2 \perp X_3 | X_1$ implies $(X_1, X_2) \perp X_3$.

(c). Show that $X_1 \perp X_3 | X_2, X_1 \perp X_3$ implies that either $X_1 \perp (X_2, X_3)$ or $(X_1, X_2) \perp X_3$.  

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Problem 0.4. Consider the graph


(b). Compute the ideal of the parametrization $I_G$ and the global Markov ideal $I_{\text{global}(G)}$ if the random variable $X \sim \mathcal{N}(0, \Sigma)$ is Gaussian.

The Macaulay2 package "GraphicalModels" might be useful.

Problem 0.5. Consider the graph

(a). Compute the global Markov statements for this DAG.


(c). Compute the ideal of the parametrization $I_G$ and the global Markov ideal $I_{\text{global}(G)}$ if the random variable $X \sim \mathcal{N}(0, \Sigma)$ is Gaussian.

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Problem 0.6. Let $X \sim \mathcal{N}(\mu, \Sigma)$ be a Gaussian random vector, and let $G = (V, E)$ be a DAG.

(a). Assume that $X$ satisfies the directed global Markov property with respect to $G$.

1. Show that $X$ satisfies the \textit{directed local Markov property} with respect to to $G$, i.e. for every $v \in V$,

$$X_v \independent X_{\text{nd}(v) \setminus \text{pa}(v)}|X_{\text{pa}(v)}.$$ 

Here nd($v$) is the set of non-descendants of $v$, i.e. all vertices to which there isn’t a directed path from $v$, and pa($v$) is the set of parents of $v$, i.e. all vertices $u$ such that there is an edge $u \rightarrow v$. 

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2. Now, define the *residuals*

\[ \epsilon_i := X_i - \Sigma_{i,\text{pa}(i)}(\Sigma_{\text{pa}(i)},\text{pa}(i))^{-1}X_{\text{pa}(i)}. \]

Show that they are Gaussian random variables and are pairwise independent.

(b). Show that if there exist \( \lambda_{ij} \in \mathbb{R} \) for all edges \( (i,j) \in E \) and independent Gaussian random variables \( \epsilon_1, \ldots, \epsilon_n \) such that

\[ X_i = \sum_{j \in \text{pa}(i)} \lambda_{ij} X_j + \epsilon_j, \]

then \( X \) satisfies the directed local Markov property with respect to \( G \).

**Problem 0.7.** Classify the Markov equivalence classes of DAGs on 4 vertices.

**Problem 0.8.** Let \( G = (V,D,B) \) be an acyclic mixed graph and let \( X \) be a Gaussian random vector with covariance matrix

\[ \Sigma = (I - \Lambda)^{-T} \Omega (I - \Lambda)^{-1}, \]

where \( \Lambda \in \mathbb{R}^D, \Omega \in PD(B) \).

(a). For a directed path \( \pi = u_0 \to u_1 \to \cdots \to u_k \), the *path monomial* \( m_\pi \) is defined as

\[ m_\pi = \lambda_{u_0 u_1} \lambda_{u_1 u_2} \cdots \lambda_{u_{k-1} u_k}. \]

Show that the \( i,j \)-th entry of \( (I - \Lambda)^{-1} \) equals

\[ ((I - \Lambda)^{-1})_{i,j} = \sum_{\text{directed paths } \pi \text{ from } i \text{ to } j} m_\pi. \]

(b). For the following graph

\[ 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \]

compute \( (I - \Lambda)^{-1} \) using part (a).

(c). A *trek* between two vertices \( i \) and \( j \) in a mixed graph \( G \) has the form

1. \( i = u_k \leftarrow u_{k-1} \cdots \leftarrow u_0 \to \cdots \to v_{\ell-1} \to v_{\ell} = j, \) or
2. \( i = u_k \leftarrow u_{k-1} \cdots \leftarrow u_0 \leftrightarrow v_0 \to \cdots \to v_{\ell-1} \to v_{\ell} = j \)
In both cases $k, \ell$ are nonnegative integers. For a trek $\tau$ the *trek monomial* $m_{\tau}$ is:

$$m_{\tau} = \lambda_{u_{k-1}u_k} \cdots \lambda_{u_0u_1} \omega_{u_0u_0} \lambda_{u_0v_1} \cdots \lambda_{v_{\ell-1}v_\ell}$$

if the trek is of type 1, and

$$m_{\tau} = \lambda_{u_{k-1}u_k} \cdots \lambda_{u_0u_1} \omega_{u_0v_0} \lambda_{v_0v_1} \cdots \lambda_{v_{\ell-1}v_\ell}$$

if the trek is of type 2.

Show that the $i, j$-th entry of the covariance matrix $\Sigma = (I - \Lambda)^{-T} \Omega (I - \Lambda)^{-1}$ equals

$$\Sigma_{i,j} = \sum_{\text{treks } \tau \text{ between } i \text{ and } j} m_{\tau}.$$  

(d). For the graph from part (b). compute $\Sigma$ in terms of the entries $\Lambda$ and $\Omega$ using (c).

**Trek separation.** Let $G = (V, D, B)$ be a mixed graph. Let $A, B, C_A, C_B \subseteq V$. We say that $(C_A, C_B)$ *trek separates* $A$ and $B$ if every trek $\tau$ between a vertex in $A$ and a vertex in $B$ either goes through a vertex in $C_A$ on its left side or through a vertex in $C_B$ on its right side.

**Theorem 0.9** ([3]). The submatrix $\Sigma_{A,B}$ has rank at most $r$ for all $\Sigma \in \mathcal{M}_G$ if and only if there exist $C_A, C_B$ such that $(C_A, C_B)$ trek separates $A$ and $B$, and $|C_A| + |C_B| \leq r$.

**Problem 0.10.** For the following graph

compute $I_G$ and $I_{\text{global}(G)}$ using the Macaulay2 package ”GraphicalModels”. Further, compute the trek separation statements and identify the generators of $I_G$ corresponding to them.

**Open Problems.** A very good source of open problems regarding linear structural equation problems is Section 3 of [1].

**References**

