# Binomial Ideals Exercises

1. As in the lecture, let
$$\phi_A : k[p] \to k[t^{\pm}], \qquad p_i \mapsto t^{A_i}.$$
Prove that the binomials $\{p^u - p^v, u, v \in \mathbb{N}^n, Au = Av\}$ span $\ker \phi_A$ as a $k$-vector space (and thus also as an ideal).

2. Check out the Markov bases database under https://markov-bases.de. Does there exist a binary graphical model that has a minimal generator of odd degree? What kind of improvments/changes would you suggest to the authors of the database?

3. Assume $I_A = \left\langle p^{m^+} - p^{m^-} : m \in \mathcal{M} \right\rangle$ is a toric ideal for which a finite generating set is known. Describe an algorithm to find a grading matrix $A$.

4. The *counting if you don't know many numbers monoid* is $M = \{0, 1, 2, \infty\}$, with the expected addition $1 + 2 = \infty, \ldots$. Describe the monoid algebra corresponding to $M$ and present it as the quotient of a polynomial ring modulo a binomial ideal.

5. Find two binomial ideals $I_1 \subsetneq I_2$ that induce the same congruence.

6. Compute the binomial primary decomposition of $I_1 = \langle x^2 - xy, xy - y^2 \rangle \subset k[x, y]$. For $I_1$ and each component, draw the corresponding congruence on $\mathbb{N}^2$ and interpret. Do the same analysis for the ideals $I_2 = \langle y(x^2 - 1), y^2(x - 1), y^3 \rangle$ and $I_3 = \langle x^3 - 1 \rangle$.

7. Consider 5 binary random variables $O, I_1, \ldots, I_4$. These variables are shall model a probabilistic computation where $I_1, \ldots, I_4$ are inputs and $O$ is the output. For this exercise, let computation be implemented a as a joint distribution of all 5 variables. A distribution is *robust against one input failure* if the output is conditionally independent of one of the inputs given the remaining inputs: $O \perp\!\!\!\perp I_j \,\middle|\, I_{[4]\setminus\{j\}}$ , for all $j = 1, \ldots, 4$. Describe all joint distributions that have this robustness.

8. Compute binomial primary decompositions of many random binomial ideals, e.g. generated by 4 cubics, 5 quadrics, etc. What is the maximum number of components you find? Can you correlate the data of the primary decomposition with any other data about the ideal (dimension, regularity, projective dimension, Betti numbers, ...)? *This is a "playground problem". I can't say if something interesting will emerge.*