Hinčin's theorem for additive free convolutions of R-diagonal $*$-distributions

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Abstract

Hinčin proved that any limit law associated with a triangular array of uniformly infinitesimal random variables is infinitely divisible. Subsequently, an analogous result for additive free convolutions of tracial R-diagonal $*$-distributions was proved by the author. We prove an analogous result for additive free convolutions of nontracial R-diagonal $*$-distributions.

Theorem

Let $\{a_{nj} : n \in \mathbb{N}, j = 1, \ldots, k_n\}$ be an array of $*$-free R-diagonal elements in a C^*-probability space, let $b_n = \sum_{j=1}^{k_n} a_{nj}$, let μ_{nj} denote the $*$-distribution of a_{nj}, and let θ_{nj} and $\tilde{\theta}_{nj}$ stand for the distributions of $a_{nj}a_{nj}^*$ and $a_{nj}^*a_{nj}$, respectively. Suppose that

(a) the array $\{a_{nj}\}$ is infinitesimal, in the sense that for every $\varepsilon > 0$,
$\lim_{n\to\infty} \max_{1\le j\le k_n} \theta_{nj}(\{t : t > \varepsilon\}) = 0$, $\lim_{n\to\infty} \max_{1\le j\le k_n} \tilde{\theta}_{nj}(\{t : t > \varepsilon\}) = 0$;

(b) $T = \sup_{n\in\mathbb{N}} \|b_n\|^2 < +\infty$; and

(c) there exists a $*$-distribution μ of some C^*-element such that the sequence $(\mu_{b_n})_{n\in\mathbb{N}}$ converges weakly to μ.

Then μ is $\boxplus$-infinitely divisible.

R-diagonal $*$-distribution

Let $(\mathcal{A}, \tau)$ be a C^*-probability space, that is $\mathcal{A}$ is a unital C^*-algebra and τ is a state on $\mathcal{A}$. The $*$-distribution μ_a of the element $a \in \mathcal{A}$ is said to be *R-diagonal*, if its R-transform R_{a,a^*} is of the form

$$R_{a,a^*}(z, z^*) = f_a(zz^*) + g_a(z^*z)$$

where $\quad f_a(z) = \sum_{k=1}^{\infty} \alpha_k z^k$ and $g_a(z) = \sum_{k=1}^{\infty} \beta_k z^k$.

The sequences of complex coefficients $\{\alpha_k\}_{k=1}^{\infty}$ and $\{\beta_k\}_{k=1}^{\infty}$ are called the *determining sequences* of the R-diagonal element a. The power series f_a and g_a are called the *determining series* of a.

Conclusion

Every weak limit of sums of infinitesimal triangular array of R-diagonal variables is necessarily $\boxplus$-infinitely divisible.

Analogous Results

Analogous results to Hinčin's theorem were proved in [1] and [2], where the classical convolution is replaced by the additive free convolution of probability measures on the real line and the multiplicative free convolution of probability measures, respectively.

The analogous result in the tracial (generally, unbounded) case was proved in [3]. We note that the analytic structure required in this paper is different from the one in the tracial case, because the analytic tools from the tracial case do not apply in the nontracial case.

References

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Free Probability

C^*-probability space

Let $(\mathcal{A}, \tau)$ be a *C^*-probability space*, that is $\mathcal{A}$ is a unital C^*-algebra and τ is a state on $\mathcal{A}$.
A random variable is $a \in \mathcal{A}$. Noncommutative.

$*$-distribution

The *$*$-distribution* μ_a of an element $a \in \mathcal{A}$ is the linear functional $\mu_a : \mathbb{C}\langle Z, Z^*\rangle \to \mathbb{C}$ determined by requiring that

$$\mu_a(Z^w) = \tau(a^w),$$

where we use the notation $a^w = a^{\ell_1} \cdots a^{\ell_n}$ for $w = (\ell_1, \ldots, \ell_n)$ and $\ell_i = 1$ or $*$.

Free independent

The alternating product of centred elements is centred. More precisely, $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ are *free independent*: if

$$\tau(a_1 a_2 \cdots a_n) = 0$$

whenever $a_j \in \mathcal{A}_{i_j}$, $i_j \in \{1, 2\}, i_1 \neq i_2, i_2 \neq i_3, \cdots, i_{n-1} \neq i_n$, and $\tau(a_j) = 0$, $1 \le j \le n$.

Free additive convolution

One can define the *additive free convolution* $\mu \boxplus \nu$ of two $*$-distributions μ and ν. This operation corresponds to the sum $a+b$ of two random variables a and b in the same C^*-probability space such that $\{a, a^*\}$ is free from $\{b, b^*\}$; see [5].

$\boxplus$-infinitely divisible

A $*$-distribution μ is said to be *$\boxplus$-infinitely divisible* if, for every natural number n, there exists a $*$-distribution μ_n such that

$$\mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ times}}.$$

R-transform

The R-transform of a $*$-distribution μ is a formal power series in two non-commuting indeterminates z and z^*, whose coefficients are uniquely determined by the $*$-moments $\mu(Z^w)$ of μ.

Properties for R-diagonals

There exist probability measures θ_{aa^*} and θ_{a^*a} on the positive real half-line, such that an R-diagonal $*$-distribution μ_a is completely determined by θ_{aa^*} and θ_{a^*a}. See [6].

Classical Probability

Probability space

(1)A sample space. (2)A set of events.
(3)The assignment of probabilities to the events.
A random variable X is a measurable function. (Commutative.)

Independent

Two random variables are independent if the realization of one does not affect the probability distribution of the other. If X and Y are independent random variables, then the expectation operator E has the property

$$E[XY] = E[X]E[Y].$$

Additive convolution

Denote by $\mu * \nu$, the classical convolution of two measures μ, ν on the real line. In the probabilistic terms $\mu * \nu$ is the distribution of $X+Y$, where X and Y are real independent random variables with distributions μ and ν, respectively.

$*$-infinitely divisible

A probability measure on the real line $\mathbb{R}$ is said to be $*$-infinitely divisible if there exist probability measures $\mu_1, \mu_2, \ldots$ such that for every $n \in \mathbb{N}$

$$\mu = \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n \text{ times}}.$$

Original Hinčin's theorem

Hinčin [4] showed that $*$-infinitely divisible measures are the most general distributions that can be obtained as limit laws of sums of uniformly infinitesimal independent random variables. More precisely, assume we are given an array $\{X_{ij} : i \ge 1, 1 \le j \le k_i\}$ of real random variables, and a sequence $\{c_i : i \ge 1\}$ of real numbers, such that

(i)$X_{i1}, X_{i2}, \ldots, X_{ik_i}$ are independent for every i;

(ii)$\lim_{i\to\infty} \max_{1\le j\le k_i} P(|X_{ij}| > \varepsilon) = 0$ for every $\varepsilon > 0$; and

(iii)the distributions μ_i of $c_i + \sum_{j=1}^{k_i} X_{ij}$ converge to a limit μ (a probability measure on real line) in the weak topology.

Then Hinčin proved that the measure μ must be $*$-infinitely divisible. Conversely, every $*$-infinitely divisible measure μ occurs as a weak limit of this type.