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Some polymeric fluid flows models: steady states & large-time-convergence

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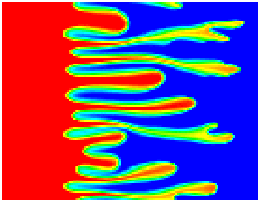
Dilute polymer suspension – applications

- multi-grade motor oil:
  polymer additives to improve/tune viscosity $\nu(p, T)$
- enhanced (tertiary) oil recovery:
  strong fingering at oil/water interface
  $\rightarrow$ front stabilization with polymer additives (increase viscosity of water)



- food industry:
  polymer additives to thicken sauces, ...

Macro Model: fluid flow

- dilute solution of polymers in homogeneous fluid
- coupled micro-macro model
- incompressible Navier-Stokes for macro flow $u(t, x)$:

$$\begin{aligned} u_t + (u \cdot \nabla_x) u &= \Delta_x u - \nabla_x p + \mathrm{div}_x \tau, \ \Omega \subset \mathbb{R}^d \\ \mathrm{div}_x u &= 0 \end{aligned}$$

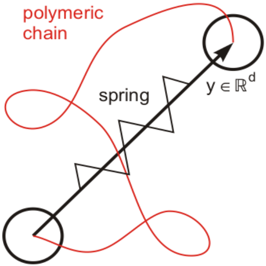
- coupling to polymer-model via extra stress tensor:

$$\tau(t, x) = \int_{\mathbb{R}^d} (y \otimes \nabla_y \Pi(y)) \psi(t, x, y) \, dy$$

(all parameters :=1)

Micro Model: polymer distribution

- dumbbell model for polymeric chains: $y \in \mathbb{R}^d$ ... extension, orientation



- micro-distribution (probility densitity in y) at each $x \in \Omega : \psi(t, x, y)$ in Fokker-Planck equ.:

$$\psi_t + \underbrace{u \cdot \nabla_x \psi}_{\text{transport in flow}} = \frac{1}{2}\mathrm{div}_y\Big([\underbrace{\nabla_y \Pi(y)}_{\text{spring force in dumbbells}} \; -2 \underbrace{(\nabla_x \otimes u)^T \cdot y}_{\text{drag force of inhom. flow field } u}]\psi\Big) + \frac{1}{2}\Delta_y \psi$$

Results from [Jourdain-LeBris-Lelièvre-Otto, ARMA 2006]

linear FP: $\psi_t = \frac{1}{2}\text{div}_y([\nabla_y \Pi(y) - 2\kappa y]\psi + \nabla\psi)$, $y \in \mathbb{R}^d$; const: $\kappa \in \mathbb{R}^{d\times d}$

1. Hookean, $\Pi = \frac{1}{2}|y|^2$:

Theorem

κ symmetric with $\lambda_j(\kappa) < \frac{1}{2}$, or κ anti-symmetric:
$\Rightarrow \exists!$ steady state ψ_∞; exp. convergence of $\psi(t)$, $t \to \infty$; general κ open !

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3. coupled nonlin. model ($\to$ log. relative entropy of $\psi + \|u - u_\infty\|^2_{L^2}$)

Theorem

FENE, if $|\kappa^s| < \frac{1}{2}$, $Tr\,\kappa = 0, [\kappa, \kappa^T]$ small (κ from BC/u_∞-steady state)
$\Rightarrow$ exp. convergence of $(u, \psi) \overset{t\to\infty}{\longrightarrow} (u_\infty, \psi_\infty)$;   Hookean open !

Outline:

1. linear Fokker-Planck: steady state & large time convergence (entropy method)
   a) non-symmetric Fokker-Planck: steady state, entropy decay
   b) Hookean dumbbells
   c) FENE dumbbells [=finite extensibility nonlinear elasticity]
2. coupled micro-macro model (Hookean)

1. Linear Fokker-Planck equ.

assume: given homogeneous flow with $u(x) = \kappa x, \kappa \in \mathbb{R}^{d\times d}$

$\rightarrow$ FP equ. for dumbbell distribution $\psi(t,y)$ in micro-variable $y \in \mathbb{R}^d$

$$\begin{cases} \psi_t = L\psi := \frac{1}{2}\text{div}_y([\underbrace{\nabla_y \Pi(y) - 2\kappa y}_{\text{not gradient!}}]\psi + \nabla_y \psi) \\ \psi(0,y) = \psi_0(y) \geq 0 \end{cases}$$

$$\psi(t,y) \geq 0, \quad \int_{\mathbb{R}^d} \psi(t,y)\, dy = \int_{\mathbb{R}^d} \psi_0 \, dy = 1$$

a) non-symmetric Fokker-Planck equ.

symmetric FP:

$$\begin{aligned} \psi_t &= L\psi = \mathrm{div}(\nabla A(y)\psi + \nabla\psi) = \mathrm{div}(\psi_\infty \nabla \frac{\psi}{\psi_\infty}); \\ \psi_\infty &= \mathrm{e}^{-A} \text{ ... steady state, normalized} \\ L & \text{ ... symmetric in } L^2(\psi_\infty^{-1}\, dy) \end{aligned}$$

relative entropy: $\mathrm{e}\,(\psi|\psi_\infty) := \int_{\mathbb{R}^d} \psi(y) \ln \frac{\psi}{\psi_\infty}\, dy \geq 0$

entropy dissipation: $\frac{d}{dt}\mathrm{e}\,(\psi(t)|\psi_\infty) = -\frac{1}{2}\int \left|\nabla \frac{\psi(t)}{\psi_\infty}\right|^2 \frac{\psi_\infty^2}{\psi(t)}\, dy \leq 0$

If A uniformly convex, $\frac{\partial^2 A}{\partial y^2} \geq \lambda I$ (Bakry-Emery condition):
$\Rightarrow$ exp. decay of relative entropy $\mathrm{e}\,(\psi(t)|\psi_\infty)$ with $\mathrm{e}^{-\lambda t}$ (at least)

a) non-symmetric Fokker-Planck equ.

non-symmetric FP

$$\psi_t = L\psi = \mathrm{div}([\nabla A + \vec{F}]\psi + \nabla\psi)$$

$\vec{F}$ given with "orthogonality condition" $\mathrm{div}_y(\vec{F}\psi_\infty) = 0 \ \forall y$

$\Rightarrow \quad \psi_\infty = e^{-A}$ still steady state

$\Rightarrow \quad$ same entropy dissipation, same decay estimate as for symm. FP

$L^{as}\psi := \mathrm{div}(\vec{F}\psi) \cdots$ skew-symmetric in $L^2(\psi_\infty^{-1}\, dy)$

$\longrightarrow$ decomposition of $L = L^s + L^{as}$

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general situation:

$$\psi_t = \mathrm{div}(\vec{G}\psi + \nabla\psi)$$

$\rightarrow$ find decomposition $\vec{G} = \nabla A + \vec{F},\ \mathrm{div}(\psi_\infty \vec{F}) = 0,\ \psi_\infty = \mathrm{e}^{-A};$

is equivalent to find ψ_∞ (similar to Helmholtz-Hodge decomposition).

Remark for Hookean $A = |y|^2/2$: $\mathrm{div}(\vec{F}) = 0$ (a-posteriori)

b) polymer model with Hookean dumbbells: $\Pi(y)=\frac{|y|^2}{2}$

$$\psi_t = L\psi = \frac{1}{2}\text{div}([y - 2\kappa y]\psi + \nabla\psi), \quad y \in \mathbb{R}^d, \; t \geq 0 \qquad (1)$$

Theorem (steady state [AA-Carrillo-Manzini, 2010])

Let $-(I - 2\kappa)$ *be stable (otherwise no confinement pot.), i.e.* $Re\lambda_j(\kappa) < \frac{1}{2}$*:*

$\Rightarrow \exists!$ *normalized steady state of* (1)*:*

$$\psi_\infty(y) = c \; exp(-\frac{1}{2} y^T \Sigma^{-1} y),$$

$$0 < \Sigma = \Sigma^T = 2\int_0^\infty e^{-(I-2\kappa)s} e^{-(I-2\kappa^T)s} \, ds$$

if κ *normal:* $\Sigma = (I - 2\kappa^s)^{-1}$. $\exists$ *standard algorithms to compute* Σ *from* κ

Proof.

Fourier transform of steady state equ: $\text{div}([y - 2\kappa y]\psi + \nabla\psi) = 0$

$$\xi^T(I - 2\kappa)\nabla_\xi\hat{\psi}(\xi) = -|\xi|^2\hat{\psi}(\xi)$$

Use ansatz $\hat{\psi}(\xi) = c\,\exp(-\frac{1}{2}\xi^T\Sigma\xi)$:

$\Rightarrow\ 0 = -(I - 2\kappa)\Sigma - \Sigma(I - 2\kappa)^T + 2I\ \ldots$ "continuous Lyapunov equ"

Since $-(I - 2\kappa)$ stable; $2I$ pos.def, symm $\Rightarrow \exists!\ \Sigma$ □

$A(y) = -\ln\psi_\infty = \frac{1}{2}y^T\Sigma^{-1}y + c\ \ldots$ uniformly convex potential of symmetric part of L in $L^2(\psi_\infty^{-1})$

Theorem (convergence in rel. entropy [ACM]:)

Let $-(I-2\kappa)$ *be stable*

$$\Rightarrow \; e(\psi(t)|\psi_\infty) \le e^{-\lambda_{min}(\Sigma^{-1})t}\, e(\psi_0|\psi_\infty), \;\; t \ge 0,$$

with $\lambda_{min} > 0$ *computable*

Proof.

entropy method [AMTU,2001], Bakry-Emery cond. for symm. part of L:

$$\frac{\partial^2 A}{\partial y^2} \ge \lambda_{\min}(\Sigma^{-1})$$

□

Rem:

decay is sharp for quadratic potentials, also for non-symmetric Fokker-Planck equ. $\forall\, \vec{F}$ (same entropy dissipation for "optimal functions")

c) FENE - dumbbells [finite extensibility nonlinear elasticity]

$$\begin{cases} \psi_t = \frac{1}{2}\operatorname{div}([\nabla\Pi - 2\kappa y]\psi + \nabla\psi), \ \underbrace{|y| < \sqrt{b}}_{\mathcal{B}}, \ b \geq 2, \ t > 0 \\ \Pi(y) = -\frac{b}{2}\ln(1 - \frac{|y|^2}{b}) \quad \rightarrow \text{r.h.s. degenerate elliptic} \\ \left(\quad \psi\big|_{\partial\mathcal{B}} = 0 \quad \right) \end{cases}$$

steady state for κ normal: $\psi_\infty(y) = ce^{-\Pi(y)+y^T\kappa^s y}$;
in general not explicit

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perturbation result:

$$L\psi = L_1\psi + L_2\psi := \frac{1}{2}\operatorname{div}\big([\nabla\Pi - 2\underbrace{\kappa_1}_{\text{normal !}} y]\psi + \nabla\psi\big) - \operatorname{div}(\kappa_2 y\psi)$$

ex.: $\kappa_1 = \kappa^s; \kappa_2 = \kappa^{as}$

$$\mu(y) := ce^{-\Pi(y)+y^T\kappa_1^s y}, \quad \int_{\mathcal{B}} \mu\, dy = 1 \qquad (\text{“prototype” steady state})$$

$L_1\mu = 0$; pos. spectral gap $\lambda_1 > 0$ ($\exists$ estimates); L_1 symm. on $L^2(\mu^{-1}\, dy)$

solve steady state equ ($\to$ eigenvalue problem for ψ):

$$L\psi = 0 \text{ with } \int_{\mathcal{B}} \psi\, dy = 1 \text{ in } \mathcal{V} = \left\{ \frac{\psi}{\mu} \in H^1(\mu\, dy) \right\} \Rightarrow \psi|_{\partial\mathcal{B}} = 0 \qquad (2)$$

alternative formulation: $\Phi := \psi - \mu \in \mathcal{V}^\perp = \{\Phi \in \mathcal{V} \mid \int \Phi\, dy = 0\} \perp \mu$

$\to$ solve for $\Phi \in \mathcal{V}^\perp$:

$$L\Phi = -L\mu = -L_2\mu$$

Theorem (FENE - small κ_2 [ACM, 2010])

if $\sqrt{b}\|\kappa_2\|_2 < \frac{\sqrt{\lambda_1}}{2} \Rightarrow \exists!$ *normalized steady state of* (2)

Proof.

quadratic form of L_1 is coercive on $\mathcal{V}^\perp$,
quadratic form of L_2 is (small) bounded perturbation □

Theorem (FENE - arbitrary κ, $\forall\, b \geq 2$ [AA-Bardos, 2010])

$$0 = L\psi = L_1\psi + L_2\psi := \frac{1}{2}\, div\big([\nabla\Pi - 2\kappa^s y]\psi + \nabla\psi\big) - div(\kappa^{as} y\psi)$$

has a normalized sol. $0 < \psi_\infty \in \mathcal{V} = \Big\{ \frac{\psi}{\mu} \in H^1(\mu\, dy) \Big\};\ \mu = c e^{-\Pi(y) + y^T \kappa^s y}$.

Proof.

- for large $\lambda > 0$: $K_\lambda := (\lambda - L)^{-1}$ compact on $\mathcal{H} := L^2(\mathcal{B}; \mu^{-1} dy)$.
- for large $\lambda > 0$: $K_\lambda : \mathcal{H}^+ := \{\psi \in \mathcal{H} \big| \psi \geq 0\} \to \mathcal{H}^+$, using:

$$K_\lambda \psi = \int_0^\infty e^{-\lambda t} \Big[\lim_{n\to\infty} \big(e^{\frac{t}{n}L_1} e^{\frac{t}{n}L_2}\big)^n \psi \Big]\, dt$$

- Krein-Rutman: spectral radius $r(K_\lambda) > 0$ is eigenvalue of K_λ; $\psi \in \mathcal{H}^+$
- $(\lambda - L)\psi = \frac{1}{r}\psi \;\Rightarrow\; (\lambda - \frac{1}{r}) \underbrace{\int_{\mathcal{B}} \psi\, dy}_{>0} = \int_{\mathcal{B}} L\psi\, dy \overset{\text{div form}}{=} 0 \;\Rightarrow\; \lambda = \frac{1}{r}$
  $\Rightarrow\; L\psi = 0$
- $\psi_\infty(y) > 0$ on $\mathcal{B}$ by min-principle on $B_\rho(0)$, $\rho < \sqrt{b}$. □

Theorem (FENE - arbitrary κ, $b = 2$ [AA-Bardos, 2011])

$0 < \psi_\infty \in \mathcal{V}$ *is unique normalized solution of* $L\psi = 0$

Proof.

- μ solves

$$L^*\psi = \frac{1}{2}\,\mathrm{div}\big(\mu\nabla\frac{\psi}{\mu}\big) + \mu\big(\nabla^\top\frac{\psi}{\mu}\cdot\kappa_2\cdot y\big) = 0 \qquad (3)$$

- every (other) solution $\psi \in \mathcal{V}$ satisfies $\frac{\psi}{\mu} \in C^\infty(\bar{\mathcal{B}})$; (degenerate ellipt.)
  $\Rightarrow\ \exists$ solution $\tilde\psi$ of (3) with $0 < C_1\mu(y) \le \tilde\psi(y) \le C_2\mu(y),\ \ y \in \mathcal{B}$
- consider $\psi(y;t) := \mu(y) - t\,\tilde\psi(y);\ t \ge 0$ ... same boundary decay!
  consider $\psi(.;t^*) \ge 0$ with $\psi(y_0;t^*) = 0,\ y_0 \in \mathcal{B},\ t^* > 0$!
  min. principle $\Rightarrow \psi(.;t^*) \equiv 0 \Rightarrow \ker(L^*) = \mathrm{span}[\mu]$
- $\dim(\ker L) = \dim(\ker L^*)$, since L has compact resolvent

□

Rem:

- [Chupin, 2009]: existence (with Leray-Schauder) & uniqueness
- here: explicit boundary behavior crucial for large-time convergence

2. Coupled micro-macro model for Hookean dumbbells: $t \longrightarrow \infty$ convergence

Navier-Stokes for $u(t,x)$ on Ω :
Choose BC $u\big|_{\partial\Omega} = \kappa x$ for some $\kappa \in \mathbb{R}^{d\times d}$ (div $u = Tr\kappa = 0$) $\Rightarrow$

$$u_\infty = \kappa x,\ \psi_\infty = c\, \mathrm{e}^{-\frac{1}{2} y^T \Sigma^{-1} y} \quad \text{is steady state.}$$

$\longrightarrow$ decay of "relative entropy" (formal; if solution $\exists$):

$$E(t) := \frac{1}{2}\int_\Omega |u(t) - u_\infty|^2\, dx + \int_\Omega \int_{\mathbb{R}^d} \psi(t) \ln \frac{\psi(t)}{\psi_\infty}\, dy\, dx$$

Theorem ([ACM])

Let $\|\kappa^s\|_2,\ \sup_t \|\underbrace{\nabla_x \otimes u^s(t,\cdot)}_{\text{deformation matrix}}\|_{L^\infty(\Omega)},\ \|\int_{\mathbb{R}^d} |y|^4 \psi_0(x,y)\,dy\|_{L^\infty(\Omega)}$ *be small;*

let $Re\,\lambda_j(\kappa) < \frac{1}{2}$.

$\Rightarrow\ E(t) \searrow 0$ *exponentially*

Proof.

- differential inequality between $\frac{dE}{dt}$, $E(t)$
- logarithmic Sobolev inequality for $\psi_\infty(y)$ ... Gaussian
- new weighted Csiszár-Kullback inequ. (was "missing link" in [JLLO])

□

Lemma ([ACM] ; weighted Csiszár-Kullback inequality)

$\psi, \phi \in L^1_+(\mathbb{R}^d)$, $\int \psi = \int \phi = 1$, $|y|^4(\psi + \phi) \in L^1(\mathbb{R}^d)$

$$\Rightarrow \; \| |y|^2(\psi - \phi)\|^2_{L^1} \le 2\, e\,(\psi|\phi) \cdot \max\left(\int |y|^4 \psi \, dy \, , \int |y|^4 \phi \, dy\right)$$

Proof.

$$e\,(\psi|\phi) = \int_{\mathbb{R}^d} \frac{\psi}{\phi} \ln \frac{\psi}{\phi} \, \phi \, dy \;\; \overset{2^{nd} \text{ Taylor}}{=} \;\; \frac{1}{2} \int_{\mathcal{A}} \frac{1}{\zeta(y)} (\psi - \phi)^2 \, dy$$

with $\mathcal{A} := \{\psi(y) \ne \phi(y)\}$; $\zeta =$ some intermediate value in (ψ, ϕ)

$$\int_{\mathcal{A}} |y|^2 |\psi - \phi| \, dy \overset{\text{Hölder}}{\le} \left(\int_{\mathcal{A}} \frac{1}{\zeta} (\psi - \phi)^2 \, dy \right)^{\frac{1}{2}} \cdot \left(\int_{\mathcal{A}} |y|^4 \zeta \, dy \right)^{\frac{1}{2}}$$

□