The Mean-Field Limit for a Regularized Vlasov-Maxwell Dynamics

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•In 1977, K. Braun and W. Hepp established the mean-field limit for a N-particle system leading to a variant of the Vlasov-Poisson model where the Coulomb potential is replaced with a smooth (at least C^2) function

•Their result was strengthened by R. Dobrushin in 1979, who estimated the rate of convergence in that limit in terms of the Monge-Kantorovich distance

Pbm: can one extend their results to the Vlasov-Maxwell system?

A short review of Dobrushin's argument

•Vorticity formulation of the 2D incompressible Euler equation

$$\partial_t \omega + \mathrm{div}_x(\omega u) = 0\,, \quad u = \nabla^\perp G \star_x \omega\,, \quad -\Delta G = \delta_0$$

•Helmholtz's N-point vortex system

$$\dot{x}_i(t) = \frac{1}{N} \sum_{1 \le j \ne i \le N} \nabla^\perp G(x_i(t) - x_j(t))$$

Problem: Does one have the mean field limit as $N \to +\infty$

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(0)} \rightharpoonup \omega^{in} \in L^1(\mathbf{R}^2) \Rightarrow \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)} \rightharpoonup \omega(t,\cdot)?$$

If so, estimate the rate of convergence.

The regularized system

•Replace G with $K \in C_b^2(\mathbf{R}^2)$ EVEN; let $t \mapsto X(t, a, \omega^{in})$ solve

$$\begin{cases} \dot{X} = \nabla^\perp K \star_x \omega(t, X)\,, \quad \omega(t, \cdot) = X(t, \cdot, \omega^{in}) \# \omega^{in} \\ X(0, a, \omega^{in}) = a \end{cases}$$

Observe that

$$x_i(t) = X\left(t, x_i(0), \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(0)}\right)$$

while the solution of the regularized 2D incompressible Euler

$$\begin{cases} \partial_t \omega + \operatorname{div}_x(\omega u) = 0\,, \quad u = \nabla^\perp K \star_x \omega \\ \omega\big|_{t=0} = \omega^{in} \end{cases}$$

is

$$\omega(t, x) = \omega^{in}(X(-t, x, \omega^{in})) \quad \text{if } \omega^{in} \in L^1(\mathbf{R}^2)$$

Dobrushin's estimate

• Therefore the mean field limit as $N \to \infty$ for the regularized system is equivalent to the continuous dependence of the solution of the regularized Euler equation in terms of its initial vorticity distribution (observation due to Braun and Hepp)

• Dobrushin obtains an estimate of the Monge-Kantorovich distance

$$\mathrm{dist}_{MK}\left(\omega(t,\cdot), \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)}\right)$$

in terms of its initial value at $t = 0$

Extension to the Vlasov-Maxwell system

Conceptual difficulties in adapting Dobrushin's program:

•Source term in Maxwell's equations is the charge+current density, i.e. a 4-vector, NOT a probability measure

•Solution of Maxwell's equation involves a retarded potential, thus one may lose the structure involving transportation of measure

•Can one preserve the Hamiltonian structure of the Vlasov-Maxwell system by regularization? at least the conservation of energy?

The relativistic Vlasov-Maxwell system

Unknown: distribution function $f \equiv f(t,x,\xi)$, $x \in \mathbf{R}^3$ position, $\xi \in \mathbf{R}^3$ momentum, and $E \equiv E(t,x)$ electric field, $B \equiv B(t,x)$ magnetic field

$$\left\{\begin{array}{l} \partial_t f + v(\xi)\cdot\nabla_x f + (E + v(\xi)\times B)\cdot\nabla_\xi f = 0 \\ \operatorname{div}_x B = 0\,, \qquad \partial_t B + \operatorname{curl}_x E = 0 \\ \operatorname{div}_x E = \rho_f\,, \qquad \partial_t E - \operatorname{curl}_x B = -j_f \end{array}\right.$$

with the notation

$$e(\xi) = \sqrt{1+|\xi|^2}\,, \qquad v(\xi) := \nabla e(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}$$

and

$$\rho_f(t,x) := \int_{\mathbf{R}^3} f(t,x,\xi)d\xi\,, \quad j_f(t,x) := \int_{\mathbf{R}^3} v(\xi)f(t,x,\xi)d\xi$$

Density of Liénard-Wiechert potential

Idea due to Bouchut-FG-Pallard (ARMA 2003)

•Define $u_f :\equiv u_f(t,x,\xi) \in \mathbf{R}$ to be the solution of

$$\square_{t,x} u_f(t,x,\xi) = f(t,x,\xi)\,, \qquad u_f\big|_{t=0} = \partial_t u_f\big|_{t=0} = 0$$

•Assume the initial data of RVM is of the form

$$B\big|_{t=0} = 0\,, \quad E\big|_{t=0} = -\nabla\phi^{in}\,, \quad \phi^{in} = (-\Delta)^{-1}\int_{\mathbf{R}^3} f^{in} d\xi$$

and define ϕ_0 to be the solution of

$$\square_{t,x}\phi_0 = 0\,, \qquad \phi_0\big|_{t=0} = \phi^{in}\,, \quad \partial_t\phi_0\big|_{t=0} = 0$$

Reconstruction of the electromagnetic field

•Knowing ϕ_0 and u_f, reconstruct the electromagnetic potential by

$$\phi_f := \phi_0 + \int_{\mathbf{R}^3} u_f d\xi \,, \quad A_f := \int_{\mathbf{R}^3} v(\xi) u_f(t,x,\xi) d\xi$$

•The electromagnetic field is given by the usual formula

$$E_f = -\partial_t A_f - \nabla_x \phi_f \,, \quad B_f = \operatorname{curl}_x A_f$$

•Because of the continuity equation

$$\partial_t \int_{\mathbf{R}^3} f d\xi + \operatorname{div}_x \int_{\mathbf{R}^3} v(\xi) f d\xi = 0$$

the electromagnetic potential (ϕ_f, A_f) satisfies the Lorentz gauge

$$\partial_t \phi_f + \operatorname{div}_x A_f = 0$$

Scalar formulation of RVM

- Therefore the RVM system is equivalent to

$$\begin{cases} \partial_t f + v(\xi)\cdot\nabla_x f + F[f]\cdot\nabla_\xi f = 0\,, \quad f\big|_{t=0} = f^{in} \\ \square_{t,x} u_f = f\,, \qquad u_f\big|_{t=0} = 0\,, \qquad \partial_t u_f\big|_{t=0} = 0 \\ \square_{t,x}\phi_0 = 0\,, \qquad \phi_0\big|_{t=0} = \phi^{in}\,, \quad \partial_t\phi_0\big|_{t=0} = 0 \\ F[f] = -\nabla_x\phi_0 - \int_{\mathbf{R}^3}(\nabla_x + v(\eta)\partial_t)u_f(t,x,\eta)d\eta \\ \qquad + \int_{\mathbf{R}^3} v(\xi)\times \mathrm{curl}_x(u_f(t,x,\eta)v(\eta))d\eta \end{cases}$$

The regularized retarded potential

•Let $\chi \in C_c^\infty(\mathbf{R}^3)$ satisfy

$$\chi(x) = \chi(-x) \geq 0\,, \quad \operatorname{supp}\chi \subset B(0,1)\,, \quad \int_{\mathbf{R}^3} \chi(x)dx = 1$$

and define the regularizing sequence $\chi_\epsilon(x) := \epsilon^{-3}\chi(x/\epsilon)$

•Forward fundamental solution of the wave equation in $\mathbf{R}_t \times \mathbf{R}^3_x$:

$$Y := \frac{\mathbf{1}_{t>0}}{4\pi t}\delta(|x| - t)$$

Set

$$Y_\epsilon := \chi_\epsilon \star_x \chi_\epsilon \star_x Y$$

The regularized RVM system

$$
\begin{cases}
\partial_t f_\epsilon + v(\xi)\cdot\nabla_x f_\epsilon + F_\epsilon[f_\epsilon]\cdot\nabla_\xi f_\epsilon = 0\,, \quad f_\epsilon\big|_{t=0} = f^{in} \\
F_\epsilon[f_\epsilon] = -\nabla_x\partial_t Y_\epsilon(t,\cdot)\star G\star\int_{\mathbf{R}^3} f^{in} d\eta \\
\qquad - \int_{\mathbf{R}^3} (\nabla_x + v(\eta)\partial_t) Y_\epsilon \star_{t,x} (\mathbf{1}_{t>0} f_\epsilon)(t,x,\eta) d\eta \\
\qquad - \int_{\mathbf{R}^3} v(\xi)\times(v(\eta)\times\nabla_x) Y_\epsilon \star_{t,x} (\mathbf{1}_{t>0} f_\epsilon)(t,x,\eta) d\eta
\end{cases}
$$

Conservation of "energy"

The regularized force is $F_\epsilon[f_\epsilon] = \chi_\epsilon \star_x E_\epsilon + v(\xi) \times (\chi_\epsilon \star_x B_\epsilon)$ with

$$\begin{cases} \operatorname{div}_x B_\epsilon = 0 & \partial_t B_\epsilon + \operatorname{curl}_x E_\epsilon = 0 \\ \operatorname{div}_x E_\epsilon = \chi_\epsilon \star_x \rho_{f_\epsilon} & \partial_t E_\epsilon - \operatorname{curl}_x B_\epsilon = -\chi_\epsilon \star_x j_{f_\epsilon} \end{cases}$$

The variation of kinetic energy is the work of the regularized force

$$\frac{d}{dt} \iint_{\mathbf{R}^3 \times \mathbf{R}^3} e(\xi) f_\epsilon dx d\xi = \int_{\mathbf{R}^3} j_{f_\epsilon} \cdot (\chi_\epsilon \star_x E_\epsilon) dx$$
$$= \int_{\mathbf{R}^3} (\chi_\epsilon \star_x j_{f_\epsilon}) \cdot E_\epsilon dx = -\frac{d}{dt} \int_{\mathbf{R}^3} \tfrac{1}{2}(|E_\epsilon|^2 + |B_\epsilon|^2) dx$$

Theorem

For any $f^{in} \in \mathcal{P}(\mathbf{R}^3_x \times \mathbf{R}^3_\xi)$ satisfying

$$\iint_{\mathbf{R}^3 \times \mathbf{R}^3} e(\xi) f^{in}(dxd\xi) < \infty$$

there exists a unique $f_\epsilon \in C(\mathbf{R}_+; w - \mathcal{P}(\mathbf{R}^3_x \times \mathbf{R}^3_\xi))$ solution of the regularized RVM systems satisfying

$$\iint_{\mathbf{R}^3 \times \mathbf{R}^3} e(\xi) f_\epsilon(t, dxd\xi) + \int_{\mathbf{R}^3} \tfrac{1}{2}(|E_\epsilon|^2 + |B_\epsilon|^2)(t,x)dx = \text{Const.}$$

•Regularization by $\chi_\epsilon \star \chi_\epsilon$ proposed by E. Horst (see also G. Rein CMS 2004)

The method of characteristics for regularized RVM 1

Denote $z=(x,\xi)$ and let $t\mapsto Z(t,z,f^{in})$ to be the solution of

$$\left\{\begin{array}{l}\dot{Z}=K[f](t,Z)\,,\\ Z\big|_{t=0}=z\end{array}\right.\qquad f(t,dz)=Z(t,\cdot,f^{in})\#f^{in}(dz)$$

where $K:\ C(\mathbf{R}_+;w-\mathcal{P}(\mathbf{R}^6))\to C_b(\mathbf{R}^6;\mathbf{R}^6)$ is the linear operator

$$K[f](t,Z)=\int_{\mathbf{R}^6}m(t,Z,\zeta)f(0,d\zeta)+\int_0^t\int_{\mathbf{R}^6}r(t,Z,\tau,\zeta)f(\tau,d\zeta)d\tau$$

Whenever $f^{in}\in L^1(\mathbf{R}^6)$, one recovers the usual formula

$$f(t,Z(t,z,f^{in}))=f^{in}(z)$$

The method of characteristics for regularized RVM 2

Here $r = (r_1, r_2)$ and $m = (m_1, m_2) \in \mathbf{R}^3 \times \mathbf{R}^3$ with

$$\left\{ \begin{aligned} m_1(t,x,\xi,y,\eta) &= \mathbf{1}_{t\geq 0} v(\xi) \\ m_2(t,x,\xi,y,\eta) &= -\partial_t \nabla_x Y_\epsilon \star_x G(t,x-y) \end{aligned} \right.$$

where G is the Green function of $-\Delta$, and

$$\left\{ \begin{aligned} r_1(t,x,\xi,y,\eta) &= 0 \\ r_2(t,x,\xi,y,\eta) &= -(\nabla_x + v(\eta)\partial_t) Y_\epsilon(t-s,x-y) \\ &\quad -v(\xi) \times (v(\eta) \times \nabla_x Y_\epsilon(t-s,x-y)) \end{aligned} \right.$$

so that

$$K[f](t,x,\xi) = (v(\xi), F_\epsilon[f](t,x,\xi))$$

The Monge-Kantorovich distance

Definition

For $\mu, \nu \in \mathcal{P}(\mathbf{R}^6)$, let $\Pi(\mu, \nu)$ be the set of $\pi \in \mathcal{P}(\mathbf{R}^6 \times \mathbf{R}^6)$ satisfying

$$\iint \phi(x)\psi(y)\pi(dxdy) = \int \phi(x)\mu(dx) \int \psi(y)\nu(dy)$$

for each $\phi, \psi \in C_b(\mathbf{R}^6)$, and

$$\begin{aligned}\mathrm{dist}_{MK}(\mu, \nu) &= \inf_{\pi \in \Pi(\mu,\nu)} \iint 1 \wedge |x - y| \pi(dxdy) \\ &= \sup_{\|\phi\|_{W^{1,\infty}(\mathbf{R}^6)} \leq 1} \left| \int \phi(z)(\mu - \nu)(dz) \right|\end{aligned}$$

Stability estimate à la Dobrushin 1

Thus $t \mapsto Z(t, z, f^{in})$ is the solution of

$$(*) \begin{cases} \dot{Z} = K[f](t, Z)\,, & f(t, dz) = Z(t, \cdot, f^{in}) \# f^{in}(dz) \\ Z\big|_{t=0} = z \end{cases}$$

where K is the integral operator with kernel

$$\mathbf{1}_{t \geq 0} m(t, z, \zeta) \delta_{\tau=0} + \mathbf{1}_{0 \leq \tau \leq t} r(t, z, \tau, \zeta)$$

satisfying, for each $t \in [0, T]$

$$\begin{aligned} &|m(t, z, \zeta_0) - m(t, z', \zeta_0')| + |r(t, z, \tau, \zeta) - r(t, z', \tau, \zeta')| \\ &\qquad \leq L_\epsilon(T)(1 \wedge |z - z'| + 1 \wedge |\zeta_0 - \zeta_0'| + 1 \wedge |\zeta - \zeta'|) \end{aligned}$$

Stability estimate à la Dobrushin 2

Theorem

For each $f^{in} \in \mathcal{P}(\mathbf{R}^6)$ and each $z \in \mathbf{R}^6$, the problem (*) has a unique C^1 solution $t \mapsto Z(t, z, f^{in})$. Moreover Z is continuous on $\mathbf{R}_+ \times \mathbf{R}^6 \times w - \mathcal{P}(\mathbf{R}^6)$ and one has

$$\text{dist}_{MK}(f(t,\cdot), g(t,\cdot)) \leq \Psi_\epsilon(t)\,\text{dist}(f^{in}, g^{in})$$

where

$$f(t,dz) = Z(t,\cdot,f^{in})\#f^{in} \text{ and } g(t,dz) = Z(t,\cdot,g^{in})\#g^{in}$$

while

$$\Psi_\epsilon(t) = (1 + tL_\epsilon(t))e^{t^2 L_\epsilon(t)}$$

The N-particle regularized RVM dynamics

Consider a gas of N particles with positions $x_i(t)$ and momenta $\xi_i(t)$, for $i = 1, \ldots, N$, satisfying

$$\begin{cases} \dot{x}_i(t) = v(\xi_i(t)) \\ \dot{\xi}_i(t) = -\frac{1}{N}\sum_{j=1}^{N} \nabla_x Y_\epsilon(t,\cdot) \star G(x_i(0) - x_j(0)) \\ -\frac{1}{N}\sum_{j=1}^{N}\int_0^t (\nabla_x + v(\xi_j(s))\partial_t) Y_\epsilon(t-s, x_i(t) - x_j(s)) ds \\ -\frac{1}{N}\sum_{j=1}^{N}\int_0^t v(\xi_i(s)) \times (v(\xi_j(s)) \times \nabla_x Y_\epsilon(t-s, x_i(t) - x_j(s))) ds \end{cases}$$

The mean field limit 1

Observe that

$$(x_i, \xi_i)(t) = Z(t, (x_i, \xi_i)(0), f^{in}) \qquad i = 1, \ldots, N$$

with

$$f^{in} = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j(0), \xi_j(0)}$$

so that

$$f(t, dxd\xi) = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j(t), \xi_j(t)}$$

The mean field limit 2

Theorem

Let $f^{in} \equiv f^{in}(x,\xi)$ be a probability density on $\mathbf{R}^6$ and set $\Omega = (\mathbf{R}^6)^{\mathbf{N}^*}$ with the Borel probability measure $\mathbf{P} = (f^{in}(x,\xi)dxd\xi)^{\otimes \mathbf{N}^*}$. For each $N \geq 1$ and each $\omega = (x_1^{in}, \xi_1^{in}, \ldots)$, let $t \mapsto (x_i^N, \xi_i^N)(t)$ be the solution of the N-body RVM dynamics with initial data $x_1^{in}, \xi_1^{in}, \ldots, x_N^{in}, \xi_N^{in}$, while f is the solution of the regularized RVM equation with $f\big|_{t=0} = f^{in}$. For $N \to \infty$ and $\mathbf{P}$-a.s. in $(x_1^{in}, \xi_1^{in}, \ldots)$

$$\operatorname{dist}_{MK}\left(f(t,x,\xi)dxd\xi, \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^N(t),\xi_i^N(t)} \right)$$

$$\leq \Psi_\epsilon(t)\operatorname{dist}_{MK}\left(f^{in}(x,\xi)dxd\xi, \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^{in},\xi_i^{in}} \right) \to 0$$

The mean field limit 3

Amplification

Moreover, under the same assumptions as above

$$\mathrm{dist}_{MK}\left(\rho_f(t,x)dx, \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i^N(t))}\right)$$

$$+\left\|j_f(t,x)dx - \frac{1}{N}\sum_{i=1}^{N} v(\xi_i^N(t))\delta_{x_i^N(t))}\right\|_{W^{-1,1}(\mathbf{R}^3)}$$

$$\leq 3\Psi_\epsilon(t)\,\mathrm{dist}_{MK}\left(f^{in}(x,\xi)dxd\xi, \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i^{in},\xi_i^{in}}\right) \to 0$$

The mean field limit 4

Amplification

...while the solution (B_ϵ, E_ϵ) of the regularized Maxwell system

$$\left\{\begin{array}{ll} \operatorname{div}_x B_\epsilon = 0 & \partial_t B_\epsilon + \operatorname{curl}_x E_\epsilon = 0 \\ \operatorname{div}_x E_\epsilon = \chi_\epsilon \star_x \rho_f & \partial_t B_\epsilon + \operatorname{curl}_x E_\epsilon = -\chi_\epsilon \star_x j_f \\ B_\epsilon\Big|_{t=0} = 0 & E_\epsilon\Big|_{t=0} = -\nabla \chi_\epsilon \star G \star \rho_{f^{in}} \end{array}\right.$$

and the corresponding fields $(B_{N,\epsilon}, E_{N,\epsilon})$ with f and f^{in} replaced with

$$f_N(t, dxd\xi) := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i^N(t), \xi_i^N(t)} \qquad f_N^{in}(dxd\xi) := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i^{in}, \xi_i^{in}}$$

The mean field limit 5

Amplification

... satisfy

$$\begin{aligned}&\|B_\epsilon - B_{N,\epsilon}\|_{L^\infty([0,T]\times\mathbf{R}^3)}\\&\quad + \|E_\epsilon - E_{N,\epsilon}\|_{L^\infty([0,T]\times\mathbf{R}^3)}\\&\qquad \leq C_\epsilon(T)\,\mathrm{dist}_{MK}\left(f^{in}(x,\xi)dxd\xi, f_N^{in}\right) \to 0\end{aligned}$$

for each $\epsilon > 0$ as $N \to \infty$, $\mathbf{P}$-a.s. in $(x_1^{in}, \xi_1^{in}, \ldots)$.

•The Braun-Hepp mean field limit of the classical N-body problem leading to the Vlasov equation with C_b^2 potential has been extended to a regularized variant of the relativistic Vlasov-Maxwell system.

•The Dobrushin estimate of the convergence rate in terms of the Monge-Kantorovich distance has been adapted to this regularized system.

•Elskens-Kießling-Ricci (CMP2009) studied a Vlasov+wave gravitational variant of the Braun-Hepp problem, but their model could not include the magnetic field.