

Existence theory for non linear transport equations

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What are non linear transport equations?

Classical, linear transport equation reads

$$\partial_t n(t, x) + \operatorname{div} (a(t, x) n(t, x)) = 0,$$

where the velocity field a is either given or is related to n through another equation.

Recently new models were introduced in various settings (traffic flow for cars or pedestrian, movement of bacteria/cells...) taking local non linear effects into account

$$\partial_t n(t, x) + \operatorname{div} (a(t, x) f(n(t, x))) = 0, \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}^d$$

The function f is given and typically decreases as the density increases.

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As before the field a is given or related to n , for instance

$$u = -\nabla\phi, \quad -\Delta\phi = g(n),$$

again for a **non linear** function g of n . See Topaz-Bertozzi, Burger, Dolak, Schmeiser and Dalibard-Perthame.

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As before the field a is given or related to n , for instance

$$u = -\nabla\phi, \quad |\phi|^2 - \Delta\phi = g(n),$$

still for a **non linear** function g of n . See DiFrancesco, Markowich, Pietschmann.

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Many other models of the same type have been derived, see Goatin, Maury, Rascle...

Existence theory

Take for example a **vanishing viscosity** approximation

$$\begin{aligned}\partial_t n_\varepsilon(t, x) + \operatorname{div}(a_\varepsilon(t, x) f(n_\varepsilon(t, x))) - \varepsilon^2 \Delta_x n_\varepsilon &= 0, \\ n_\varepsilon(t = 0, x) &= n_\varepsilon^0(x), \quad x \in \mathbb{R}^d.\end{aligned}$$

With a_ε given or coupled to n_ε , can we **pass to the limit** in this equation to obtain a weak solution?

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Easy a priori estimates

- By the maximum principle, in general all L^p bounds are propagated

$$\|n_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R}^d)} \leq C_t \|n_\varepsilon^0(\cdot)\|_{L^p(\mathbb{R}^d)}, \quad n_\varepsilon(t, \cdot) \geq 0 \text{ if } n^0 \geq 0.$$

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- Taking for example $a_\varepsilon = -\nabla \phi_\varepsilon$ with $-\delta \phi_\varepsilon = g(n)$, one has **compactness** on a_ε

$$\begin{aligned} \|\operatorname{div} a_\varepsilon\|_{L^\infty} &\leq C_g \|n_\varepsilon\|_{L^\infty}, \\ \|a_\varepsilon\|_{W^{1,p}} &\leq C_g \|n_\varepsilon\|_{L^p} \quad 1 < p < \infty. \end{aligned}$$

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How to obtain **compactness on n_ε** ?

State of the art

Several existence results are already available in **restricted settings**.

- Existence (even of strong solutions) for **short times** is relatively **easy**. It is only valid up to the first shock.
- The **1-d case** can usually be solved with **compensated compactness** or other regularizing effects.
- For some very **precise couplings** like

$$\partial_t n(t, x) - \operatorname{div}(\nabla \phi(t, x) f(n(t, x))) = 0, \quad -\Delta \phi = n,$$

gradient flows techniques can be used (see Dolbeaut, Maury, Santambrogio).

- For the case

$$\partial_t n(t, x) - \operatorname{div}(\nabla \phi(t, x) f(n(t, x))) = 0, \quad -\Delta \phi = g(n),$$

compactness has been proved by Dalibard-Perthame using the rigidity in the kinetic formulation.

General assumptions

Let us summarize the **general framework** we would like to work in. Consider for n_ε^0 compact and uniformly bounded in $L^1 \cap L^\infty$ a sequence of solutions to

$$\begin{aligned} \partial_t n_\varepsilon(t, x) + \operatorname{div}(a_\varepsilon(t, x) f(n_\varepsilon(t, x))) - \varepsilon^2 \Delta_x n_\varepsilon &= 0, \\ n_\varepsilon(t=0, x) &= n_\varepsilon^0(x), \quad x \in \mathbb{R}^d. \end{aligned}$$

Assume

$$\begin{aligned} \exists p > 1, \quad \sup_\varepsilon \sup_{t \in [0, T]} \|a_\varepsilon(t, \cdot)\|_{W^{1,p}(\mathbb{R}^d)} &< \infty, \\ \sup_\varepsilon \|\operatorname{div}_x a_\varepsilon\|_{L^\infty([0, T] \times \mathbb{R}^d)} &< \infty. \end{aligned}$$

And

$$\begin{cases} \operatorname{div}_x a_\varepsilon = d_\varepsilon + r_\varepsilon & \text{with } d_\varepsilon \text{ compact and} \\ \exists C > 0, \text{ s.t. } \forall \varepsilon > 0, \forall x, y, \\ |r_\varepsilon(x) - r_\varepsilon(y)| \leq C |n_\varepsilon(t, x) - n_\varepsilon(t, y)|. \end{cases}$$

The main result

Theorem

Under the previous assumptions, the sequence n_ε is **compact** in L^1_{loc} and if $a_\varepsilon \rightarrow a$, it converges to the **entropy solution** to

$$\begin{aligned}\partial_t n(t, x) + \operatorname{div}(a(t, x) f(n(t, x))) &= 0, \\ n(t = 0, x) &= \lim n_\varepsilon^0(x), \quad x \in \mathbb{R}^d.\end{aligned}$$

Entropy solution means the usual: For any χ convex and smooth, there exists η s.t.

$$\partial_t \chi(n(t, x)) + \operatorname{div}(a(t, x) \eta(n(t, x))) \leq 0.$$

A transparent example

The case where $a(t, x) = a(x)$ with $\operatorname{div} a = 0$ is straightforward. Define the flow

$$\partial_t X(t, x) = a(X(t, x)), \quad X(0, x) = x,$$

and introduce

$$g(t, s, x) = n(t, X(s, x)).$$

Then g solves the $1 - d$ scalar conservation law

$$\partial_t g + \partial_s(f(g)) = 0.$$

Compactness follows from the usual theory for linear transport equations and scalar conservation laws (with some **complication** with the **Laplacian**).

The difficulty in the general case

The problem is that the **usual methods** to obtain compactness either for linear transport equations or SCL are **not compatible**. For instance, for scalar conservation laws,

- **Compensated compactness** or other regularizing effects require a **non degeneracy** assumption.

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- **Compensated compactness** or other regularizing effects require a **non degeneracy** assumption. Here, except for $d = 1$, the equation is always degenerate as the derivative is taken in the direction of a .

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- **Compensated compactness** or other regularizing effects require a **non degeneracy** assumption.
- The propagation of **BV bounds** is **false** for linear transport equations.
- In general using **L^1 contraction** does not work because $n(t, x + h)$ is not a solution if n is.

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- DiPerna-Lions **renormalized solutions** require to pass to the **limit** and use the **uniqueness** of the limit. Here we do not know how to pass to the limit...
- Methods based on the **characteristics** (Crippa-DeLellis, Hauray-LeBris-Lions) do not work well with **shocks**.

A new norm

Define

$$\|n\|_{h,p}^p = \int_{\mathbb{R}^{2d}} \frac{|n(x) - n(y)|^p}{(h + |x - y|)^d} \mathbb{I}_{|x-y| \leq 1} dx dy.$$

This is a sort of extension of usual Sobolev or Besov norms to the case of 0 derivative. Recall for example that for $s > 0$

$$\|n\|_{H^s}^2 = \int_{\mathbb{R}^{2d}} \frac{|n(x) - n(y)|^2}{|x - y|^{d+2s}} dx dy.$$

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However the previous equality fails for $s = 0$ and the corresponding norm is **stronger than L^2** and is enough to control the high frequencies.

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Proposition

Assume that for a sequence n_k uniformly bounded in L^p

$$\limsup_k |\log h|^{-1} \|n_k\|_{h,p}^p \longrightarrow 0 \quad \text{as } h \rightarrow 0,$$

then the sequence n_k is *compact* in L_{loc}^p .

A quantitative estimate

It is possible to obtain **explicit bounds** on $\|n_\varepsilon\|_{h,1}$ for the problem we consider.

Proposition

Under all previous assumptions and in particular $a_\varepsilon \in W^{1,p}$ uniformly, one has for a constant C independent of ε and h

$$\begin{aligned} \|n_\varepsilon\|_{h,1} &\leq C |\log h|^{1/\bar{p}} + C \frac{\varepsilon^2}{h^2} \\ &\quad + C \|n_\varepsilon^0\|_{h,1} + C \|d_\varepsilon\|_{h,1}, \end{aligned}$$

with $\bar{p} = \min(2, p)$.

Conclusion and perspectives

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- Technical improvements are probably possible: the rate is likely not optimal, the case where $a_\varepsilon \in BV$ ($W^{1,1}$ can be obtained by interpolation).

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- Developing good numerical schemes for those models is crucial.

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- Technical improvements are probably possible: the rate is likely not optimal, the case where $a_\varepsilon \in BV$ ($W^{1,1}$ can be obtained by interpolation).
- Developing good numerical schemes for those models is crucial.
- The condition $\operatorname{div} a_\varepsilon \in L^\infty$ is in principle not necessary for some models. The non linearity lets us keep L^∞ bounds on the density n_ε .

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