Orbital stability of spherical galactic models

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ICERM, Providence, September 18-24, 2011

The Vlasov Poisson Model

- Gravitational Vlasov-Poisson system (VP) :

$$\partial_t f + v\cdot\nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0, \qquad f(t=0,x,v) = f_0(x,v)$$

$$\phi_f(t,x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\rho_f(t,y)}{|x-y|}dy, \quad \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v)dv.$$

(equivalently : the Poisson equation is $\Delta\phi_f = \rho_f$)

- Extensions :
  - Relativistic version : replace v by $\frac{v}{\sqrt{1+|v|^2}}$.
  - Vlasov-Manev : replace the interaction potential $\frac{1}{|x-y|}$ by $\frac{1}{|x-y|^2}$.
  - Dimension $N=4$.

Basic properties

- Conservation of the energy : $\mathcal{H}(f) = E_{kin}(f) - E_{pot}(f)$

$$E_{kin}(f) = \int_{\mathbb{R}^6} |v|^2 f dx dv, \qquad E_{pot}(f) = \int_{\mathbb{R}^3} |\nabla_x \phi_f|^2 dx$$

- Conservation of the Casimir functionals $\displaystyle\int_{\mathbb{R}^6} G(f) dx dv$.
- Scaling symmetry : f solution $\Longrightarrow$ $\frac{\mu}{\lambda^2} f\left(\frac{t}{\lambda\mu}, \frac{x}{\lambda}, \mu v\right)$ solution too.
- In the case of spherically symmetric solutions $f(t, |x|, |v|, x \cdot v)$, the angular momentum $\int_{\mathbb{R}^6} |x \times v|^2 f dx dv$ is also conserved.

Cauchy Theory

A key interpolation inequality :

$$E_{pot}(f) \leq C E_{kin}(f)^a \; \|f\|_{L^1}^b \; \|f\|_{L^p}^c \quad \text{for} \quad p \geq p_{crit}$$

Existence of solution as long as the kinetic energy is controlled.

- Classical VP in dimension 3, $a = 1/2$. Global existence : Arsen'ev 1975, Illner-Neunzert 1979, Horst-Hunze 1984, Diperna-Lions 1988, Pfaffelmoser 1992, Lions-Perthame 1991, Schaeffer 1991, ...
- Relativistic VP in dimension 3, $a = 1$. Blow-up in finite time is possible : Glassey-Schaeffer 1986.
- Vlasov-Manev in dimension 3 or classical VP in dimension 4, $a = 1$. Blow-up in finite time is possible : Bobylev-Dukes-Illner-Victory 1997.

A class of steady states

$$v \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0.$$

- Spherical galactic models :

$$f(x, v) = F\left(\frac{|v|^2}{2} + \phi_f(x)\right).$$

- Anisotropic models :

$$f(x, v) = F\left(\frac{|v|^2}{2} + \phi_f(x), |x \times v|^2\right).$$

In fact, for spherical geometry, $f(t, x, v) = f(t, |x|, |v|, x \cdot v)$, The Jeans theorem ensures that all spherically symmetric steady states are of this form (Batt-Faltenbacher-Horst 86) : Heuristically : three variables + one equation $\Longrightarrow$ two free variables, and we already know two invariants.

Stability of steady states (part 1)

Problem 1 : Spherical perturbations

All anisotropic steady states

$$Q(x,v) = F\left(\frac{|v|^2}{2} + \phi_Q(x), |x \times v|^2\right)$$

which are decreasing functions of the microscopic energy are stable under spherical perturbations.

- Proved in ML-Méhats-Raphaël 2009 arXiv : 0904.2443, Comm. Math. Phys, 2011.
- Optimal : Non spherical perturbations may give instabilities, Binney-Tremaine.

Stability of steady states (part 2)

Problem 2 : General perturbations

All spherically symmetric steady states depending on the energy only

$$Q(x,v) = F\left(\frac{|v|^2}{2} + \phi_Q(x)\right)$$

which are decreasing functions of the microscopic energy are stable under general perturbations.

Our objective here is to **prove this part 2 of the conjecture**, by extending the method introduced in the anisotropic case. [1]

1. ML, F. Méhats, P. Raphaël : to appear in Inventiones Math., 2011.

Linear Stability

- **Linear stability** of spherical decreasing models $Q(x, v) = F(\frac{|v|^2}{2} + \phi_Q(x))$, has been shown in the physics literature : Antonov 1961, Doremus-Baumann-Feix 1973, Sygnet-Des Forets-Lachieze-Rey-Pellat 1984, Kandrup-Sygnet 85, Perez-Aly 1996, ...
- The "Energy-Casimir" method : Linearize this functional around Q (Similar to Arnold's method in fluid mechanics.)

$$H_C(f) = \mathcal{H}(f) + \|j(f)\|_{L^1}, \qquad F = j'^{-1}.$$

Antonov's inequality : for spherically symmetric functions,

$$\int \frac{g^2}{|F'|} dxdv - \int |\nabla \phi_g|^2 dx \geq \int \frac{\xi^2}{|F'|} \frac{\phi_Q'(r)}{r} dxdv,$$

with $\quad g = v \cdot \nabla_x \xi - \nabla_x \phi_Q \cdot \nabla_v \xi.$

Non linear orbital stability : variational approaches

Construct ground states by minimizing the Hamiltonian under constraint(s).

- One-constraint problem : Wolansky (1999), Guo, Rein (1999), Schaeffer (2004), Dolbeault ...

$$\inf \quad \{\mathcal{H}(f), \quad \|f\|_{L^1} + \beta \|j(f)\|_{L^1} = M\}$$

- Two-constraints : ML-Méhats-Raphaël, (CRAS 2005, ARMA 2008), Sanchez-Soler (ARMA 2007) for polytropes $j(t) = t^p$.

$$\inf \quad \{\mathcal{H}(f), \quad \|f\|_{L^1} = M_1, \quad \|j(f)\|_{L^1} = M_j\}$$

- Prove compactness of minimizing sequences : Concentration-compactness principle, Lions 1983.

- Uniqueness of the minimizer is classically needed : Fails here.

Variational approaches

- Only uniqueness under equimeasurability constraints is needed. Provided for free by the flow. (ML-Méhats-Raphaël, CPDE 2008).
- Inequivalence of ensembles holds in many cases (the one-contraint problems cover less cases than the two-constraints problems).

Drawbacks of these approaches :

- Assumption on j : $j(t) \geq Ct^p, \quad p > 9/7$. Slow growing j's are not reached. Ex : the King model : $j(t) = (1+t)\ln(1+t) - t$.
- Not all the physical steady states are ground states (numerical simulation).

Non variational approaches

- A non variational strategy has been used recently by Guo, Rein and Lin to prove the stability of the King model *under spherically symmetric perturbations*.
- It relies on linearization techniques and a crucial coercivity property of the linearized system, directly deduced from Antonov's inequality.

The starting idea of the new approach

Minimize the Hamiltonian under an infinite number of constraints (all the Casimir functionals)

$$\inf \quad \{\mathcal{H}(f), \quad f \in \mathrm{Eq}(Q)\}$$

Still not sufficient to cover all the physical set of steady states. The new approach can be interpreted in two different ways :

- We will reach all the steady states $F\left(\frac{|v|^2}{2} + \phi_f(x)\right)$, where F is decreasing, as local minimizers of this problem. Local compactness of minimizing sequences, induces stability under general perturbations.
- Quantitative control of the distance of f from Q by the Hamiltonian and the set of all Casimirs.

Assumption on Q and equimeasurability

(i) $Q(x,v) = F\left(\frac{v^2}{2} + \phi_f(x)\right)$ is a continuous, compactly supported, steady state of VP ;

(ii) there exists e_0 such that, on $]-\infty, e_0[$, $F(e) > 0$ is C^1 with $F' < 0$ and, on $[e_0, +\infty[$, $F(e) = 0$.

Includes polytropes, King model, but many other models ...

Definition : consider the set $\mathrm{Eq}(Q)$ of nonnegative functions $f \in L^1 \cap L^\infty$ that are equimeasurable with Q :

$$\int G(f(x,v))dxdv = \int G(Q(x,v))dxdv, \qquad \forall G$$

Statement of the main results

Theorem 1. Local quantitative control of the potential.
For all $f \in \mathcal{E}$ such that ϕ_f is in the vicinity of ϕ_Q, we have

$$\inf_{z\in\mathbb{R}^3} \|\nabla\phi_f - \nabla\phi_Q(\cdot - z)\|_{L^2}^2 \leq C\left[\mathcal{H}(f) - \mathcal{H}(Q) + \|f^* - Q^*\|_{L^1}\right]$$

Theorem 2. Quantitative control of the full distribution function.
For all $f \in \mathcal{E}$, there holds :

$$\|f - Q\|_{L^1} \leq \|f^* - Q^*\|_{L^1} + C_Q\left[\mathcal{H}(f) - \mathcal{H}(Q) + \|f^* - Q^*\|_{L^1} + \frac{1}{2}\|\nabla\phi_f - \nabla\phi_Q\|_{L^2}^2\right]^{3/16}.$$

Statement of the main results

Corollary. Orbital stability of Q.

For all $\varepsilon > 0$, for all $M > 0$, there exists $\eta > 0$ such that the following holds true. Let $f_0 \in L^1 \cap L^\infty$, with $f_0 \geq 0$ and $|v|^2 f_0 \in L^1$, be such that

$$\|f_0 - Q\|_{L^1} < \eta, \quad \mathcal{H}(f_0) \leq \mathcal{H}(Q) + \eta \quad \|f_0\|_{L^\infty} < \|Q\|_{L^\infty} + M,$$

then there exists a translation shift $z(t)$ such that the corresponding weak solution $f(t)$ to VP satisfies : $\forall t \geq 0$,

$$\|(1 + |v|^2)(f(t,x,v) - Q(x - z(t), v)\|_{L^1(\mathbb{R}^6)} < \varepsilon.$$

The main ingredients of the proof

(i) Symmetric rearrangements with respect to the microscopic energy
➠ Generalization of the classical Schwarz symmetric rearrangement

(ii) Monotonicity of the Hamiltonian with respect to this rearrangement ➠ Reduction to a variational problem on the potential only
Similar ideas can be found in the physics literature : Lynden-Bell 69, Wiechen-Ziegler-Schindler 88, Aly 89.

(iii) Coercivity of the Hessian of this new functional : based on a Poincaré inequality ➠ *This is a new Antonov-like inequality*

(iv) Quantitative control of the full distribution function in terms f^* and ϕ_f.

Symmetric rearrangement, monotonicity of the Hamiltonian

The standard Schwarz symmetrization.
Let $f \in L^1(\mathbb{R}^3)$, then there exists a unique nonincreasing function $f^* \in L^1(\mathbb{R}_+)$ such that $f^*(|x|)$ is equimeasurable with f.

Generalization : rearrangement with respect to the microscopic energy.
Let $\phi(x)$ be the Poisson potential of a distribution function.
Let $f \in L^1 \cap L^\infty(\mathbb{R}^6)$, then we may define its rearrangement $f^{*\phi}$ which is :

- a nonincreasing function of $\frac{|v|^2}{2} + \phi(x)$;
- such that $f^{*\phi} \in \mathrm{Eq}(f)$.

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Identification : The "fundamental identity on the steady state"

Our assumptions on Q are enough to show that

$$Q^{*\phi_Q} = Q$$

Explicit construction of $f^{*\phi}$

Step 1. We first define the standard Schwarz rearrangement $s \mapsto f^*(s)$ $(s \in \mathbb{R}^+)$ of the function $f(x,v)$.

Remark : $g \in \mathrm{Eq}(f) \iff g^*(s) = f^*(s)$.

Step 2. We introduce the dependence in the microscopic energy :

$$f^{*\phi}(x,v) := f^*\left(a_\phi\left(\frac{|v|^2}{2} + \phi(x)\right)\right) 1_{\frac{|v|^2}{2}+\phi(x)<0}$$

where a_ϕ is a Jacobian function defined by

$$\begin{aligned} a_\phi(e) &= \text{meas}\left\{(x,v) \in \mathbb{R}^6 : \frac{|v|^2}{2} + \phi(x) < e\right\} \\ &= \frac{8\pi\sqrt{2}}{3}\int_0^{+\infty} (e-\phi(x))_+^{3/2}\,dx \end{aligned}$$

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The key monotonicity property

Lemma. *Let f be a distribution function and ϕ_f its Poisson potential. Then*

$$\mathcal{H}(f) \geq \mathcal{H}(f^{*\phi_f}).$$

Proof. Denote $\widehat{f} = f^{*\phi_f}$. We have the decomposition

$$\mathcal{H}(f) = \mathcal{H}(\widehat{f}) + \frac{1}{2}|\nabla\phi_f - \nabla\phi_{\widehat{f}}|^2_{L^2} + \int \left(\frac{|v|^2}{2} + \phi_f\right)(f - \widehat{f})dxdv.$$

By construction of $f^{*\phi_f}$, the green term is nonnegative. This is reminiscent from the following property of the standard Schwarz symmetrization :

$$\int_{\mathbb{R}^3} |x|f(x)dx \geq \int_{\mathbb{R}^3} |x|f^*(x)dx.$$

□

Reduction to a problem on the potential

Consider now our minimization problem in f with equimeasurability constraint :

$$\min \{\mathcal{H}(f) : f \in \mathrm{Eq}(Q)\}.$$

If $f \in \mathrm{Eq}(Q)$, we have $f^* = Q^*$ and the previous monotonicity equality reads

$$\mathcal{H}(f) = \mathcal{H}(Q^{*\phi_f}) + \frac{1}{2}|\nabla\phi_f - \nabla\phi_{Q^{*\phi_f}}|_{L^2}^2 + \int \left(\frac{|v|^2}{2} + \phi_f\right)(f - Q^{*\phi_f})dxdv.$$

Two facts :

- The red term only depends on the potential ϕ_f.
- The green term is nonnegative and vanishes when $f = Q^{*\phi_f}$.

This leads us to consider the following minimization problem for the potential only :

$$\min \mathcal{J}(\phi), \quad \text{with} \quad \mathcal{J}(\phi) = \mathcal{H}(Q^{*\phi}) + \frac{1}{2}\|\nabla\phi - \nabla\phi_{Q^{*\phi}}\|_{L^2}^2$$

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The reduced variational problem on ϕ

We now consider the variational problem $\min \mathcal{J}(\phi)$, with

$$\mathcal{J}(\phi) = \int \left(\frac{|v|^2}{2} + \phi(x) \right) Q^{*\phi}(x,v)\,dxdv + \frac{1}{2}\|\nabla\phi\|_{L^2}^2$$

$$Q^{*\phi}(x,v) = Q^*\left(a_\phi \left(\frac{|v|^2}{2} + \phi(x) \right) \right)$$

Proposition. *The quantity $\mathcal{J}(\phi) - \mathcal{J}(\phi_Q)$ controls the distance of ϕ to the manifold of translated Poisson fields $\mathcal{M} = \left\{ \phi_Q(\cdot + z), \quad z \in \mathbb{R}^3 \right\}$: in the vicinity of $\mathcal{M}$, we have*

$$\mathcal{J}(\phi) - \mathcal{J}(\phi_Q) \geq C \inf_{z\in\mathbb{R}^3} \|\nabla\phi - \nabla\phi_Q(\cdot - z)\|_{L^2}^2 \quad \textit{with } C > 0.$$

Proof. Based on a Taylor expansion. We differentiate twice the functional $\mathcal{J}$ with respect to ϕ and study the Hessian : it is nonnegative, and coercive on spherical functions.

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First variation of $\mathcal{J}$

We prove that

$$D\mathcal{J}(\phi)(h) = -\int_{\mathbb{R}^3} (\nabla \phi_{Q^{*\phi}} - \nabla \phi) \cdot \nabla h \, dx.$$

Consequence :

Since $Q = Q^{*\phi_Q}$ ("fundamental identity of the steady state"), we have $\phi_{Q^{*\phi_Q}} = \phi_Q$, hence $D\mathcal{J}(\phi_Q)(h) \equiv 0$.

This shows that :

ϕ_Q is a critical point of the function $\mathcal{J}$

.

Second variation of $\mathcal{J}$

We prove that

$$D^2\mathcal{J}(\phi_Q)(h,h) = \int |\nabla h|^2 dx - \int (h(x) - \Pi h(x,v))^2 \left| F'(e(x,v)) \right| dxdv$$

with $e(x,v) = \frac{|v|^2}{2} + \phi_Q(x)$.
Here Π is the projector on functions of $e(x,v)$:

$$\Pi h(x,v) = \frac{\int \left(\frac{|v|^2}{2} + \phi(x) - \phi_Q(y)\right)_+^{1/2} h(y)dy}{\int \left(\frac{|v|^2}{2} + \phi(x) - \phi_Q(y)\right)^{1/2} dy}.$$

Crucial : the quadratic form $D^2\mathcal{J}(\phi_Q)(h,h)$ is coercive, up to the degeneracy induced by the translational invariance.

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Crucial : the quadratic form $D^2\mathcal{J}(\phi_Q)(h,h)$ is coercive, up to the degeneracy induced by the translational invariance.

Coercivity of the Hessian (1/2)

Step 1. On radial functions we have a Poincaré inequality :

$$\int |\nabla h|^2 dx \geq \int (h(x) - \Pi h(x,v))^2 \left| F'(e(x,v)) \right| \, dxdv$$

Remark. This is a new Antonov-like inequality.
Proof. Adaptation of Hörmander's approach for sharp weighted Poincaré inequalities

$$\int_{\mathbb{R}^N} \left(f - \frac{\int f d\mu}{\int d\mu} \right)^2 d\mu \leq C \int_{\mathbb{R}^N} |\nabla f|^2 d\mu, \quad d\mu = e^{-V(x)} dx,$$

under the convexity assumption $\nabla^2 V \geq C_0$.

Coercivity of the Hessian (2/2)

Step 2. Decomposing $h = h_0 + h_1$ with h_0 radial and h_1 orthogonal to radial functions, one gets

$$D^2\mathcal{J}(\phi_Q)(h,h) = D^2\mathcal{J}(\phi_Q)(h_0,h_0) + (\mathcal{L}h_1, h_1)$$

with $\mathcal{L} = -\Delta + V_Q$ and $V_Q(x) = \int |F'(e)| dv$. Moreover, by translation invariance,

$$\mathcal{L}(\nabla \phi_Q) = 0.$$

Hence, since ϕ_Q is monotone increasing, a standard argument based on an expansion on spherical harmonics yields that $\mathcal{L}$ restricted on $(\dot{H}^1_{rad})^{\perp}$ is nonnegative and admits $\mathrm{Span}\,\{\partial_{x_i}\phi_Q,\ 1 \le i \le 3\}$ as kernel.

Summary

- For all $f \in \mathcal{E}$

$$\mathcal{H}(f) - \mathcal{H}(Q) + |\phi_Q(0)| \|f^* - Q^*\|_{L^1} \geq \mathcal{J}(\phi_f) - \mathcal{J}(\phi_Q)$$

- If the potential is in the vicinity of ϕ_Q, then

$$\mathcal{H}(f) - \mathcal{H}(Q) + |\phi_Q(0)| \|f^* - Q^*\|_{L^1} \geq C \inf_{z \in \mathbb{R}^3} \|\nabla \phi - \nabla \phi_Q(\cdot - z)\|_{L^2}^2$$

Control of $\|f - Q\|_{L^1}$

Theorem 2 states the control of $\|f - Q\|_{L^1}$ in terms of $\mathcal{H}(f) - \mathcal{H}(Q)$ and $\|f^* - Q^*\|_{L^1}$:

$$\|f - Q\|_{L^1} \leq \|f^* - Q^*\|_{L^1} + C_Q \left[\mathcal{H}(f) - \mathcal{H}(Q) + \|f^* - Q^*\|_{L^1} + \frac{1}{2}\|\nabla\phi_f - \nabla\phi_Q\|_{L^2}^2\right]^{3/16}.$$

Tool : Refined bathtub principle and inequalities. We found out a corrective term in the bathtub inequality for rearrangements.

Quantitative control - Summary

- Local quantitative control of the potential

$$\inf_{z\in\mathbb{R}^3} \|\nabla\phi - \nabla\phi_Q(\cdot - z)\|_{L^2}^2 \leq C\left[\mathcal{H}(f) - \mathcal{H}(Q) + \|f^* - Q^*\|_{L^1}\right]$$

- Quantitative control of the full distribution function

  For all $f \in \mathcal{E}$, there holds :

$$\|f - Q\|_{L^1} \leq \|f^* - Q^*\|_{L^1} + C_Q\left[\mathcal{H}(f) - \mathcal{H}(Q) + \|f^* - Q^*\|_{L^1} + \frac{1}{2}\|\nabla\phi_f - \nabla\phi_Q\|_{L^2}^2\right]^{3/16}.$$

Conclusion

- Parallel with incompressible fluids is possible (2D incompressible Euler) : Our method should cover the result by Lin (2004), but a complete stability result is not clear yet.
- Coupling Vlasov with general relativity.
- Does Landau damping (some asymptotic stability) hold for these considered steady states ?

$$\|\nabla_x \phi - \nabla_x \phi_Q(x + z(t))\|_{L^2} \to 0 \qquad as \quad t \to \infty.$$

THANK YOU !