Bicycle Polygons, Solitons, and the Darboux Transform

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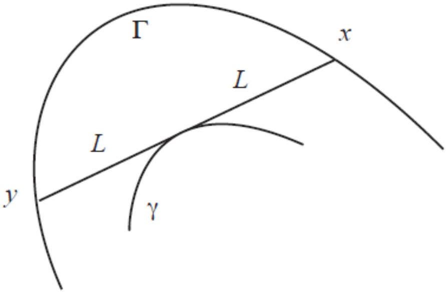
Motivation behind the Problem

The Adventure of the Priory School by Arthur Conan Doyle

Figure: S. Tabachnikov, 2006



Definitions (S.Tabachnikov 2006)

Definition (Continuous Bicycle Curve)

Two planar curves $\gamma(t)$ and $\Gamma(t)$ are said to be bicycle curves if there exists $L \in \mathbb{R}$ such that $\Gamma(t) = \gamma(t) + L\gamma'(t)$.

Definition (Ambiguous Bicycle Curves)

We call the bicycle curve ambiguous if $\Gamma(t) = \gamma(t) - L\gamma'(t)$.

Definition (Rotation Number)

If (γ, Γ) are ambiguous bicycle curves parametrized so that the perimeter of Γ is 2π then we define the flotation number

$$\rho = \frac{|\overset{\frown}{xy}|}{2\pi}.$$

Examples

Figure: F. Wegner, 2007

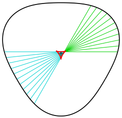


Fig. 10 $m/n = 1/3$, $\epsilon = 0.1$

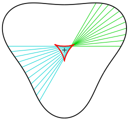


Fig. 11 $m/n = 1/3$, $\epsilon = 0.2$

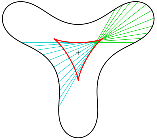


Fig. 12 $m/n = 1/3$, $\epsilon = 0.5$

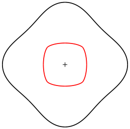


Fig. 13 $m/n = 1/4$, $\epsilon = 0.1$

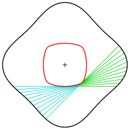


Fig.* **14** $m/n = 1/4$, $\epsilon = 0.1$

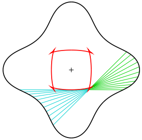


Fig. 15 $m/n = 1/4$, $\epsilon = 0.2$

Connection to Flotation Problem

For n odd the innermost envelope corresponds to density $\rho = 1/2$.

Question (Ulam's Problem 19 from the Scottish Book)

Is the sphere the only solid of uniform density that floats in equilibrium in every position?

Answer

- $\rho = 1/2$ *there exist counterexamples (Auerbach, 1938)*
- *The circle is the only such shape when* $\rho = 1/5, 2/5$ *(J. Bracho, L. Montejano, and D. Oliveros, 2001) and* $\rho = 1/4$ *(S. Tabachnikov, 2006)*

Definition

A Zindler carousel with n chairs is a system of curves $\{\beta_i\}_{i=1}^n$ such that all β_i are parametrizations of the same curve and

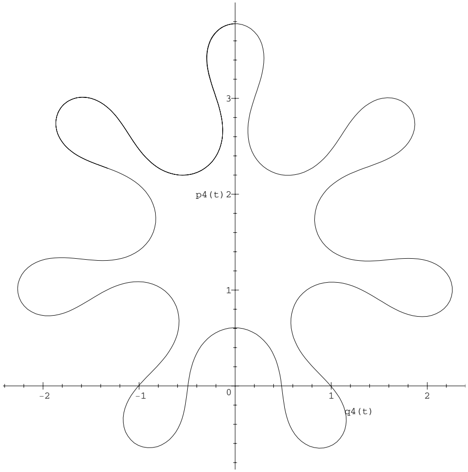
- $$\beta_{i+n}(t) = \beta_i(t)$$
- $$\beta_{i+1}(t) - \beta_i(t) = constant$$
- $m_i(t) = \frac{\beta_i(t)+\beta_{i+1}(t)}{2}$ has tangent parallel to $\beta_{i+1}(t) - \beta_i(t)$.

Theorem (J. Bracho, L. Montejano, D. Oliveros)

Zindler carousels solve the flotation problem.

Figure: Zindler Carousel with Five Chairs

14 J. BRACHO, L. MONTEJANO AND D. OLIVEROS



Definition (Bicycle Polygon)

An n-gon with vertices given by $\{V_i\}_{i=0}^{n}$ is called an (n,k)-bicycle polygon if it is equilateral and all the k diagonals are of equal length. The ratio $\frac{k}{n}$ will be called the rotation number. It is closed if $V_0 = V_n$ and periodic if $V_0 + \vec{e_1} = V_n$.

Given a regular n-gon we can construct a $(2n, k)$ bicycle polygon for any odd $k < n$ (S. Tabachnikov, 2006).

Question

Are there other constructions?

Theorem (S. Tabachnikov, 2006)

Every convex (n, k)-bicycle polygon is regular if

- $k = 2$
- n *odd and* $k = 3$
- $n = 2k + 1$
- $n = 3k$

Proposition (T., A.)

If n is odd then every convex $(4n, n)$-bicycle polygon is regular. If n is even then all odd vertices lie equally-spaced on one circle and all even vertices lie on a concentric circle at equal-spacing.

Proof.

Darboux transform of a Rhombus. □

Darboux Transformation

Definition (Darboux Transformation)

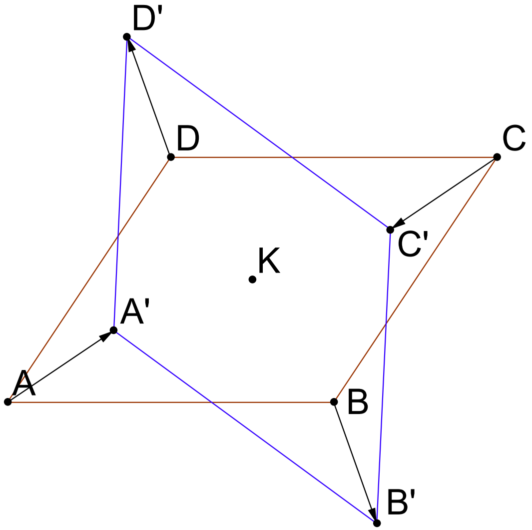
We say that two polygons P and Q are Darboux transforms of each other with parameter l if each quadrilateral $P_i Q_i P_{i+1} Q_{i+1}$ is an isosceles trapezoid with side length l:

$$|P_i Q_i| = |P_{i+1} Q_{i+1}| = l \text{ and } P_i Q_{i+1} \parallel Q_i P_{i+1}.$$

Given a vector $\vec{ov}$ and a vector $\vec{ou}$, a calculation shows that the image of u under the trapezoidal rule is given by

$$D_{\vec{ov}}(u) = \frac{|\vec{ov}|^2 - |\vec{ou}|^2}{|\vec{uv}|^2}\vec{uv}.$$

Darboux Transform of a Parallelogram

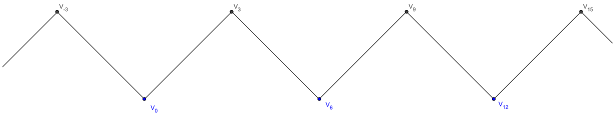


Definition (Trapezoidal Condition)

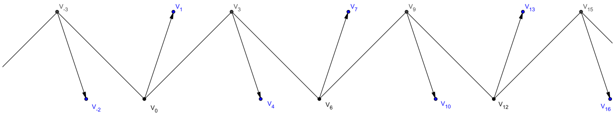
We will say that a discrete (n, k)-path satisfies the trapezoidal condition if $V_i V_{i+k+1} \parallel V_{i+1} V_{i+k}$ for each $i \in \mathbb{N}$..

A discrete (n, k) path satisfying the trapezoidal condition may be interpreted in terms of the Darboux Transform.

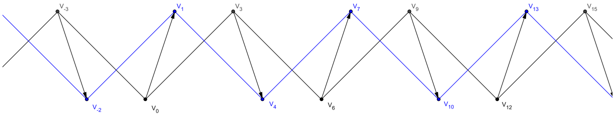
- Consider a periodic framework of n bars of equal length joined at their endpoints $E_0, E_1, ..., E_n$ in a linear arrangement. - Apply the Darboux Transformation k times so that E_i becomes E_{i+1}. - If the linkage satisfies these conditions, we may label each E_i as V_{ik} and $D_l(V_{ik+m}) = V_{ik+m+1}$ for $m = 0, 1, ..., k-1$. The result is an (n, k)-path satisfying the trapezoidal condition.

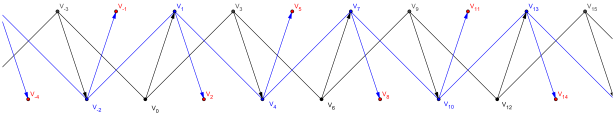


v_{-3}
v_3
v_9
v_{15}
v_0
v_6
v_{12}

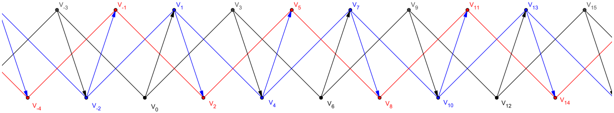


v_{-3}
v_1
v_3
v_7
v_9
v_{13}
v_{15}
v_{-2}
v_0
v_4
v_6
v_{10}
v_{12}
v_{16}

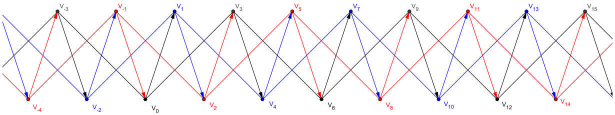

V-3
V1
V3
V7
V9
V13
V15
V-2
V0
V4
V6
V10
V12



V-3
V-1
V1
V3
V5
V7
V9
V11
V13
V15
V-4
V-2
V0
V2
V4
V6
V8
V10
V12
V14

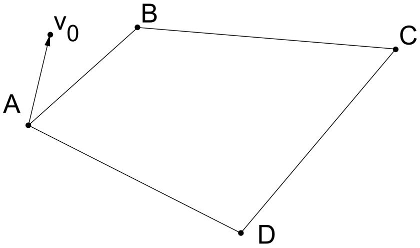


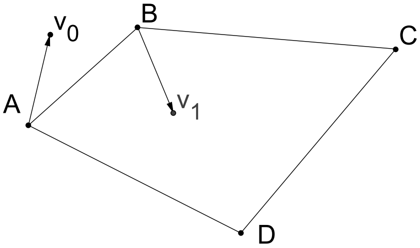
v_{-3}
v_{-1}
v_1
v_3
v_5
v_7
v_9
v_{11}
v_{13}
v_{15}
v_{-4}
v_{-2}
v_0
v_2
v_4
v_6
v_8
v_{10}
v_{12}
v_{14}

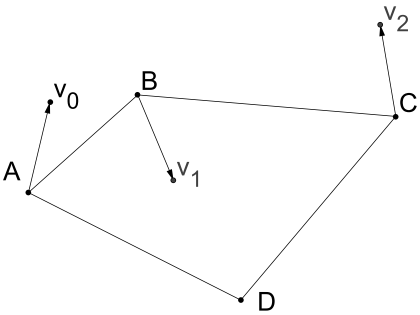


V-3
V-1
V1
V3
V5
V7
V9
V11
V13
V15
V-4
V-2
V0
V2
V4
V6
V8
V10
V12
V14

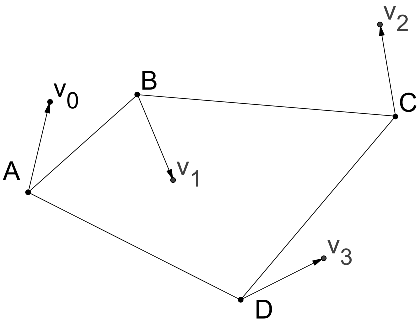
Consider a polygonal line P with vertices $V_0, V_1, ..., V_{n-1}$. Let v_0 be a vector with its origin at V_0. Having a vector v_i at vertex V_i, we obtain a vertex v_{i+1} of the same length at V_{i+1} via the trapezoidal rule.


v0
B
C
A
D

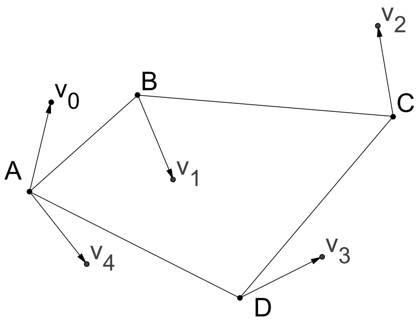

v0
B
C
A
v1
D



v2
B
v0
C
A
v1
D



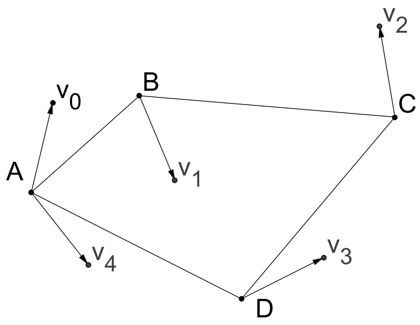
v_2
B
v_0
C
A
v_1
v_3
D


v2
v0
B
C
A
v1
v3
v4
D

For a fixed length of v_0, we may view the map taking v_0 to v_j as a self-map of the circle of radius $|v_0| = |v_j|$ by identifying the circle at V_0 with circle at V_j.

Definition ((Monodromy for the Darboux Transformationn)

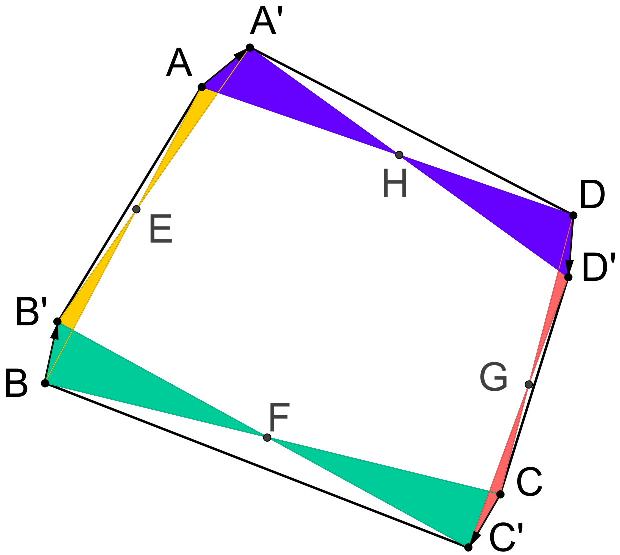
In the n-periodic and closed cases, the monodromy map is the map acting on the identified circles at V_0 and V_n which takes v_0 to v_n.



Theorem

The Darboux transformation is area and perimeter preserving on closed polygons.

Darboux Transformations Preserve Area



(Darboux Transform of a Triangle) Let $T = V_0 V_1 V_2$ be a triangle and let d be the diameter of its circumcircle. The monodromy map is hyperbolic on $(0, d)$, parabolic at d and elliptic on (d, ∞). The Darboux transforms of T are rotations of T inside of its circumcircle and these are precisely those induced by vectors from V_0 to the circumcircle.

Corollary

The Darboux transformation of a rhombus R is a rhombus congruent to R.

Proof.

Let s be the side length of R and let α be one of its angles. The area of R is

$$s^2 \sin \alpha.$$

Therefore $D(R)$ must have angle α or $\pi - \alpha$. In both cases $D(R)$ is congruent to R. □

Open Problems

- Darboux transform for quadrilaterals
- Dynamics of the Darboux transform
- Connections to group theory?
- Interpretation of the Darboux transform in hyperbolic and spherical geometry
- Vector flow for odd linkages

Continuous Vector Flow on 3-Linkages