On Bicycle Uni-track Path Efficiency: Bicycle “Unicycle” Paths

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1 Abstract

A traveling bicycle will usually leave two paths behind it: one created by the back wheel and one created by the front wheel. This leads to an intriguing question: when do the front wheel and the back wheel have the same path? An obvious answer would be, when the bicycle travels in a straight line, but indeed there are more complex paths that create a “unicycle” path with a bicycle.

This paper will explore these kinds of paths in which the front wheel path and the back wheel path coincide. What are the properties of such of path? First, we shall establish a discrete, non-smooth version of our question in which the path is composed of straight segments and circle arcs. Using this construction, we prove theorems about the complexity of the path including exponential growth of both the length of the path and its total absolute curvature.

In the case when the path is a smooth C^∞ curve, we prove linear increase of the number of points of inflection.

2 Introduction

In his paper on the subject, David Finn begins by recounting a scene from a Sherlock Holmes mystery in which Watson and Holmes are attempting to figure out which direction a bicycle was traveling by examining the tracks left on the ground [1]. Finn goes on to imagine a similar scenario in which the famous crime fighting team are presented with only one tire track. We will call this type of path a unicycle path. Excluding the case in which the perpetrator is actually riding a unicycle, what can be deduced about the path?

We can begin with simple properties of a bicycle to establish our mathematical model of its motion. The front and back wheel each touch the ground at exactly one point and the frame connecting them is rigid. The rear wheel remains in line with the frame, but the front wheel may turn. We assume that the ground is flat and analyze the tire path entirely in $\mathbb{R}^2$.

In our construction the bicycle is of unit length so the distance between the front and back wheel is always 1. We initially position the bicycle on the horizontal axis with the rear wheel sitting at the origin and the front wheel located at $(1, 0)$.

Definition The *seed curve*, γ_0, of a unicycle path is the curve from which the rest of the path is constructed. It is a piecewise smooth curve with endpoints $(0, 0)$ and $(1, 0)$ and with infinite zero derivative at each of these end points. It may self intersect and is not necessarily a graph. When the bicycle begins its motion, the front wheel moves so that the back wheel traces along this seed curve. See *Figure 1.*

The new path created by the front wheel as the back wheel traces the seed is called the first *iteration*, γ_1, of the seed curve. By requiring γ_0 to have an infinite derivative of zero at both $\gamma_0(0)$ and $\gamma_0(1)$, we force the curve to be flat at each end point, lying tangent to the x-axis at the beginning and end of every iteration. At the end of the first iteration, the front wheel of the bicycle will sit at $(2, 0)$ and the back wheel at $(1, 0)$. This process continues such that in the second iteration, γ_2, the front wheel moves so that the back wheel will trace the first iteration and so on. After the n^{th} iteration the bicycle will lie with its front wheel at

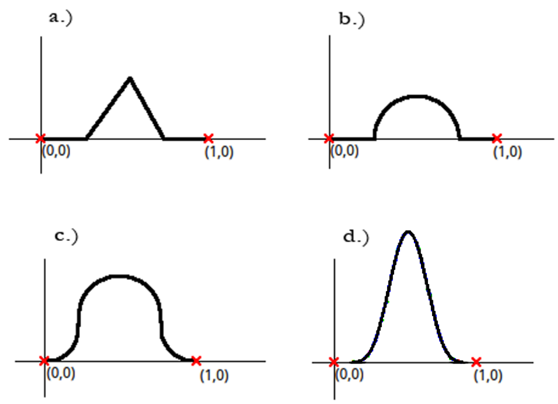

Figure 1: Possible seed curves. Note that of these, only *d.)* creates a path that is infinitely smooth

$(n+1,0)$ and back wheel at $(n,0)$ and the tangent to the path at these end points is $y=0$. It is important to remember that in this construction the back wheel is dictating the movement of the bicycle and not vice versa. With this construction, the unicycle path is necessarily continuous and infinite.

In his paper, Finn draws upon the obvious fact that for any seed curve $\gamma_0 \in C^\infty$, the next iteration, γ_1, is equal to $\gamma_1 = \gamma_0 + \gamma_0'/|\gamma_0'|$ [1]. This comes from the fact that the new curve is always a unit away along the tangent from the old curve. In this paper, for simplicity, we will assume that $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots$ are all parameterized by arc length. We will call the union of the seed curve and all subsequent iterations, i.e. the entire bicycle path, $\Gamma = \bigcup_{n=0}^{\infty} \gamma_n$.

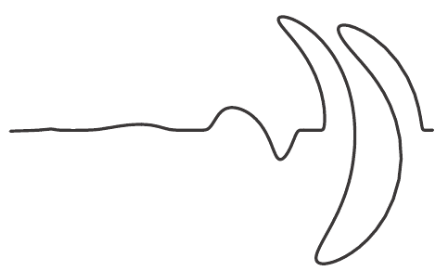

Figure 2: Finn's construction of a smooth bicycle unicycle track [1]

Little is known about bicycle curves constructed from continuously differentiable seed curves. A straight line segment seed curve will force the bicycle to ride a straight line infinitely, but it can be seen in *Figure 2* that even a small perturbation in a straight line segment seed curve causes the resulting bicycle paths to behave wildly in only a few iterations.

In [2], Mark Levi and Sergei Tabachnikov conjecture that the amplitude of Γ for $\gamma_0 \in C^\infty$ is unbounded; in other words unless γ_0 is a straight line segment, Γ is not contained in any horizontal strip. In the same paper, Levi and Tabachnikov also conjecture that unless $\gamma_0 \in C^\infty$ is a straight line segment, Γ is not a the graph of a function, fails to be an embedded curve, and, furthermore, that the curvature of Γ is unbounded. The truth of these conjectures is not yet known and they prove difficult to study. However, Levi and

Tabachnikov successfully prove the following propositions about the complexity of bicycle curves in C^∞.

Proposition 1 (a) (**Levi, Tabachnikov**) *Denote by $Z(\gamma_n)$ the number of intersection points of the curve γ_n with the x-axis (excluding the end points of the iteration); assume that $Z(\gamma_n)$ is finite. One has $Z(\gamma_{n+1}) > Z(\gamma_n)$ for any non-trivial smooth bicycle path [2].*

(b) *Denote by $E(\gamma_n)$ the number of local extrema of the curve γ_n; assume that $E(\gamma_n)$ is finite. One has $E(\gamma_{n+1}) > E(\gamma_n)$ for any non-trivial smooth bicycle path [2].*

While difficult to analyze in the smooth case, it is possible to prove Levi and Tabachnikov's conjectures about the growth of the curve's amplitude through iterations and the curve's failure to be an embedding if we examine instead *discrete* unicycle paths. In the discrete path, γ_0 is constructed strictly out of connected line segments and circle arcs (see *Figure 1 a-c*). We will now go through the construction of discrete bicycle paths and the proofs of these conjecture.

3 Discrete Model

In the discrete model of the unicycle path, the seed curve is composed of straight line segments and circle arcs. If the rear wheel encounters a corner in the seed curve (or elsewhere in the path), then the bicycle will pivot with the rear wheel fixed at the corner until the bike is tangent to the new path. This creates a unit circle arc centered at the corner with angle equal to the difference between the tangents at the corner. We will assume that the front wheel spins the least distance necessary to reach the appropriate tangent to continue its path. We can also assume that the front wheel will always turn less than π because a turn of π would imply that the bicycle is tangent to the new arc in the opposite direction; this is equivalent to the bicycle moving backwards which we will disallow.

The discrete model seed curve must begin and end with straight line segments so that it necessarily has infinite derivatives of zero at both end points. We will consider the horizontal straight line seed curve to be the trivial case since in this case the bike will never deviate from the x-axis.

We will first state a few simple facts about the movement of the bicycle in this construction.

Lemma 1 (a) *When the rear wheel traces a directed circle arc (clockwise or counterclockwise) of radius r, the front wheel traces a concentric circle arc of radius $\sqrt{r^2+1}$ in the same direction. See* Figure 3.

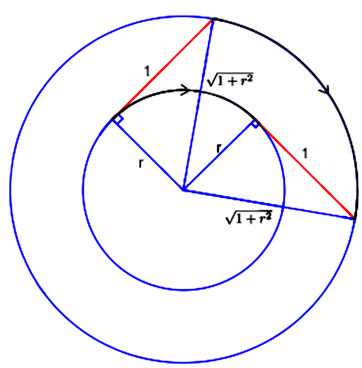

Figure 3: Unit length bicycle draws concentric circle arc path

The new arc begins where the tangent of the starting point of the old arc intersects the new concentric circle. And likewise, the new arc ends where the tangent of the ending point of the old arc intersects the new concentric circle. This is obvious from simple geometry and *Figure 3*

(b) *When the rear wheel traces a portion of the straight line $y = mx + b$, the front wheel traces the same line $y = mx + b$ exactly one unit ahead of the back wheel.*

This statement implies that any straight line propagates on a ray to infinity. See *Figure 4*

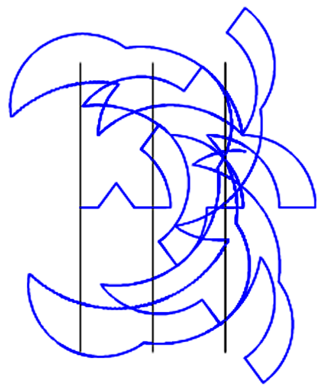

Figure 4: Straight line propagation

Lemma 1 and the construction of the path show us that a straight line segment produces another straight line segment in the next iteration, a circle arc produces a concentric circle arc, and a corner produces a unit circle arc. We can draw a few simple consequences.

Lemma 2 (a) *If the seed curve of a unicycle path is made up of line segments and circle arcs then all subsequent iterations will be made only of straight lines and circle arcs.*

(b) *The number of arcs and line segments in a given iteration is equal to the number of arcs and line segments in the previous iteration* plus *the number of corners in the previous iteration.*

Lemma 3 *About every corner in the unicycle path the bike will trace infinite concentric circle arcs of growing radius.*

Proof This is due to the simple fact that a corner creates a circle arc in the next iteration and that this circle arc creates a concentric circle arc in the next iteration and every following iteration. *Lemma 1 (a)* states that these concentric circle arcs must grow in radius. $\square$

Proposition 2 *If a seed curve is contained in a circle of radius r centered at the origin, then the n^{th} iteration is contained in a circle of radius $r + n$ centered at the origin.*

Proof At any time the rear wheel sits on the bicycle path, the front wheel must lie on the unit circle centered at the location of the rear wheel. If the seed curve is completely contained in a disc of radius r centered at the origin, then all such unit circles centered about a point on the seed curve will be contained within a disc of radius $r + 1$ centered at the origin. This implies that the first iteration is contained in the disc of radius $r + 1$ centered about the origin. This same argument applies to any following iteration. Inductively, it follows the n^{th} iteration is contained in a disc of radius $r + n$ centered about the origin. See *Figure 5.* $\square$

Proposition 3 *If γ_0 is a discrete seed curve, even if γ_0 is in C^1, then γ_1 and all subsequent iterations are necessarily C^0 and not C^1. The consecutive arcs and/or line segments that make up γ_1 and all subsequent iterations necessarily meet at a corner.*

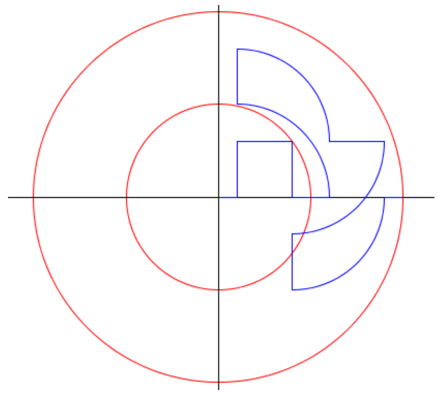

Figure 5: Linear growth rate of a disk containing a unicycle path

Proof Even if a discrete seed curve is C^1, the second derivative of the curve is discontinuous where consecutive circle arcs and/or line segments meet. Thus, the first iteration will necessarily have a corner in the unicycle path where consecutive circle arcs and/or line segments meet. See *Figure 6*.

In addition, a connection between two circle arcs and/or line segments in which the first derivative is continuous cannot arise in any *future* iterations beyond the seed curve. In other words, circle arcs and line segments will always be connected at a *corner* after the seed curve.

This is because, once a circle arc has been traced by the back wheel it hits a corner so the front wheel must pivot while the back wheel is on the corner, making a circle arc of radius 1. See *Figure 7.* This necessarily creates a new tangent at the connection point that is perpendicular to the frame of the bike (the dashed vector in *Figure 7*). In order for this new tangent to *also* be the tangent of the circle arc that was just traced by the front wheel (the dotted vector in *Figure 7*), the bike frame would have to lie on the radius of the circle arc which is impossible because the bike must be lying *tangent* to the concentric arc created by the back wheel. □

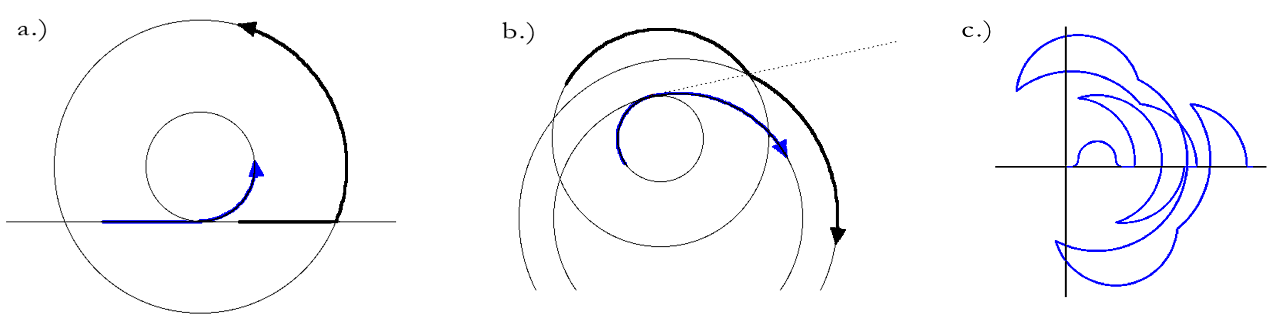

Figure 6: a.) & b.) The first iteration of a C^1 discrete seed curve, c.) The first two iterations of a seed curve with continuous first derivative.

It follows that the number of corners in a unicycle path doubles in consecutive iterations. Another obvious corollary is the following:

Corollary 3.1 *A non-trivial discrete unicycle path will have a corner on the x-axis in the first iteration of the seed curve connecting a horizontal straight line segment to either an arc or a line segment of non-zero slope.*

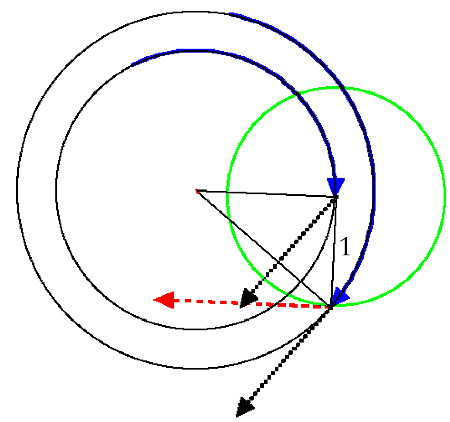

Figure 7: The front wheel must trace the unit circle but this circle is not tangent to the current dotted tangent so a new corner must be made.

As a result of the existence of this corner, we will see that all non-trivial discrete unicycle paths necessarily share a common portion of their path. We will call this portion, which is independent of the specific seed curve, the *invariant* potion of the every non-trivial discrete unicycle path. We will now describe the construction of this invariant set of arcs.

3.1 Invariant Behavior of All Discrete Unicycle Paths

We know from *Corollary 3.1* that every unicycle path Γ must contain a corner on the the horizontal axis. Let us call the first such corner x_0. When the back wheel sits at x_0, the front wheel will sit at a point on the horizontal axis, let it be called x_{11}. Then the front wheel will trace a circle arc c_0 of radius 1 in the direction that x_0 turns with length equal to the angle at x_0. We know that c_0 necessarily meets the horizontal axis with an initial tangent of $\pi/2$ from the axis. See *Figure 8*.

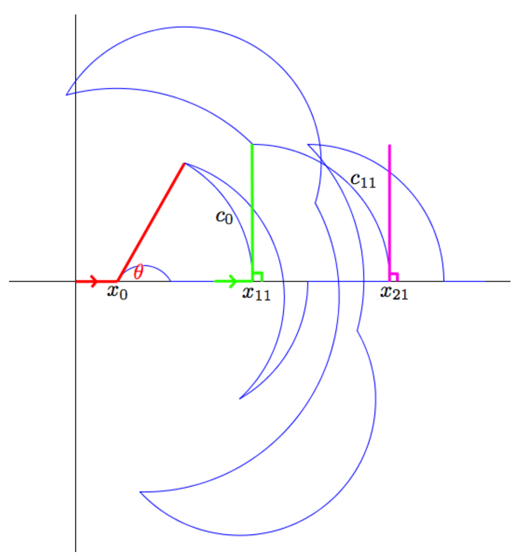

Figure 8: Construction of invariant portion in a generic non-trivial discrete bicycle path

In the next iteration when the back wheel is sitting at x_{11}, the front wheel will need to pivot to meet with this $\pi/2$ tangent, created a unit circle arc of length $\pi/2$. Let us call this new circle arc c_{11}.

This circle arc, c_{11} will appear in *every* discrete unicycle path regardless of the seed curve. In turn, this arc makes tangent that is $\pi/2$ to the axis so every future iteration will begin with a circle arc of length $\pi/2$ similar to c_{11}. We can call these arcs $c_{21}, c_{31}, c_{41}, c_{51}$, respectively. *See Figure 9.*

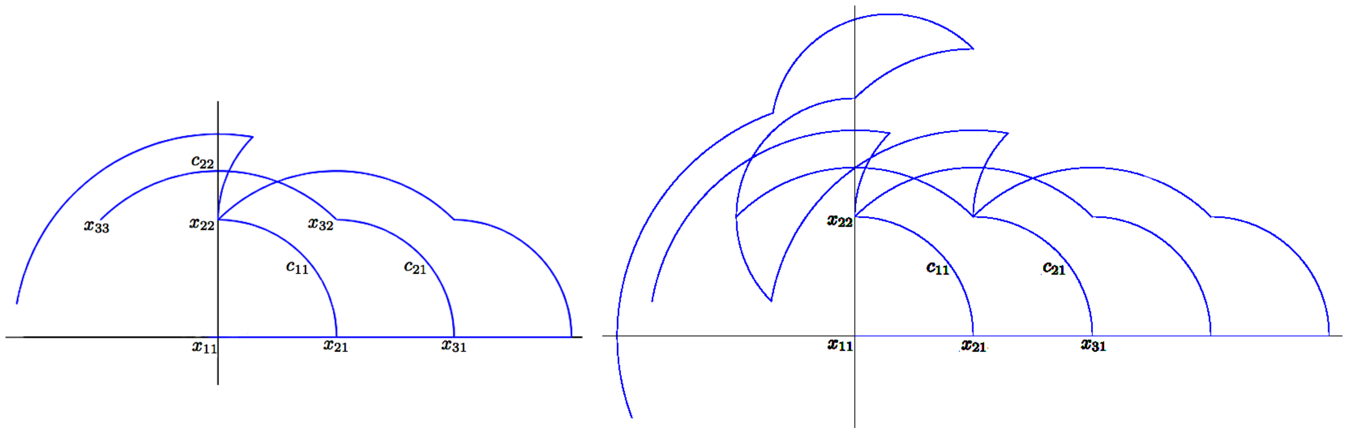

Figure 9: Invariant propagation of a corner on the x-axis

Next, every iteration that follows an iteration with an arc of $\pi/2$ must trace this arc after it has turned $\pi/2$ itself. This will result in a new circle arc of length $\pi/2$ and radius $\sqrt{2}$ that is the second arc in all of these iterations (this will not include the iteration that contains c_{11}). *See Figure 9.* We can call these new arcs c_{32}, c_{42}, c_{52}, respectively.

The next iteration must trace these new arcs and this pattern will continue ad infinitum.

Though c_0 does depend on x_0 and the seed curve, c_{11} and all the curves that are created from it are independent of this initial step. Therefore, there is an infinite pattern that is invariant in *every* non-trivial discrete unicycle path.

Definition The *invariant portion* of a discrete unicycle path is the series of arcs created through iterations of a corner that makes a $\pi/2$ tangent to the x-axis. It appears in every discrete unicycle path regardless of the seed curve.

An obvious consequence of the existence of this invariant portion of the path are the following remarks.

Remark 1 *An isometric copy of the invariant path described above, will appear about any corner in which a straight line connects to either an arc or a straight line segment with different slope.*

Remark 2*Non-trivial discrete unicycle paths cannot be an embedding, that is all non-trivial discrete unicycle paths must self-intersect.*

We can now prove our first theorem saying that non-trivial, discrete unicycle paths are unbounded in all directions.

Theorem 4 *Concerning any non-trivial discrete unicycle path,* Γ*, for any infinite ray* l *originating from a corner,* x_{11}*, in which a straight line connects to either an arc or a straight line segment with different slope, and for any* $d \in \mathbb{R}$ *there is a point* $c \in \{\Gamma \cap l\}$ *such that* $|c - x_{11}| > d$*.*

Proof To prove our claim we will examine a specific part of the invariant portion of the path: the infinite concentric circle arcs centered at corner x_{11} in *Figures 8 & 9*. By *Lemma 1* we know that the radii of the concentric circles arc will increase by $\sqrt{r^2+1}$ where r is the radius of the previous circle arc. In this case, $r = 1$ so the radii are $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}$, etc.

We can then form triangles, as in *Figure 10*, connecting x_{11} and the ends of consecutive arcs. Some simple geometry shows us that, because the bicycle is unit length and c_{11} is of length $\pi/2$, that the consecutive concentric arcs overlap and any ray emerging from x_{11} that intersects the endpoint of an arc will necessarily also intersect the following arc.

In addition, the length $\sqrt{n}$ grows unbounded as n goes to infinity and the sum of the angles at x_{11}, $\arcsin(1/\sqrt{n})$, also diverges as n goes to infinity.

Thus, for any infinite ray, l, originating at x_{11}, and for any $d \in \mathbb{R}$, there will be an intersection point, c, of the bicycle path, Γ, with ray l such that $|c - x_{11}| > d$. $\square$

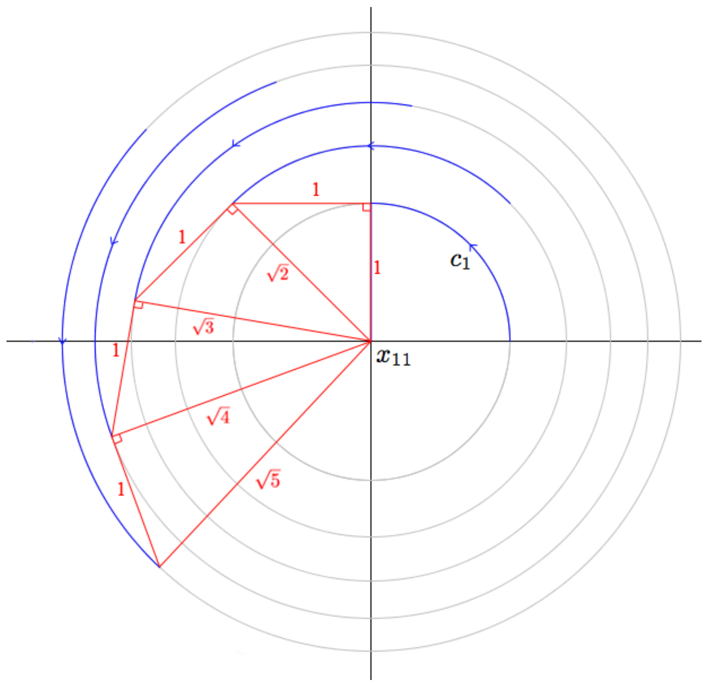

Figure 10: Invariant propagation of a corner on the x-axis

There are a number of corollaries that we can take from this theorem about the growth of a discrete unicycle path.

Corollary 4.1 *A non-trivial discrete unicycle path will cross any vertical line infinitely many times.*

Corollary 4.2 *The only unicycle path that draws a graph is the trivial path.*

Now we know that a discrete unicycle path grows infinitely in any direction. However, it is important to note that at any given moment the path is contained in a circle of known radius.

3.2 Growth of Length and Curvature in the Discrete Model

We will now use the known invariant pattern found in all discrete unicycle paths to examine the growth rate of the length of the curve and the growth of the total absolute curvature through iterations. In order to do so, we will classify and analyze the various kinds of corners that arise in the invariant path.

The total absolute curvature of a smooth curve is defined as $\int |\kappa| \, ds$. This definition works for adding up the total absolute curvature of the straight lines segments and circle arcs of a discrete path. However, the discrete model also contains corners, so we must consider the curvature of these exterior angles. We will define the total absolute curvature at each corner to be the angle that the bicycle frame sweeps through such that the bicycle lies tangent to the next arc and/or line segment. Recall that the wheel turns the shortest distance from one tangent to the next and thus every exterior angle will be less than π radians. For our purposes, the angles do not have sign.

Definition Let θ_i be the angle measure of the i^{th} corner in a discrete unicycle path. The *total absolute curvature* of a discrete unicycle path is equal to

$$\int |\kappa| \, ds + \sum_{i}^{\infty} |\theta_i|$$

Focusing on the invariant portion of the path, we see that the first type of corner that connects two circle arcs is an acute turn where both arcs have the same sign of curvature (positive or negative) and the turn is in the opposite direction of the curvature. See *Figure 11* .

Lemma 4 *An acute corner that connects two circle arcs of the same orientation but turns in the opposite direction of the curvature, produces two obtuse angles that connect arcs of opposite orientation in the next iteration.*

Proof We know from *Lemma 1* that when the back wheel is tracing an oriented circle arc, the front wheel traces a concentric circle arc in the same direction of curvature. We can easily picture where these two new arcs lie. See *Figure 11.* In this case the bike must turn in the opposite direction of the curvature which implies that the front wheel must, in a sense, go slightly backward to reach the tangent line of the next arc. This implies that front wheel must turn $\pi/2 + \theta$ radians, then draw a unit arc in the direction opposite of the curvature until it reaches the next tangent, and then turn again $\pi/2 + \phi$ to once again be going in the original direction of curvature.

This process creates three new arcs. Two arcs of the same curvature connected in the middle by a unit arc of opposite curvature. In addition, the corners will be turns of more than $\pi/2$ which makes them obtuse.

□

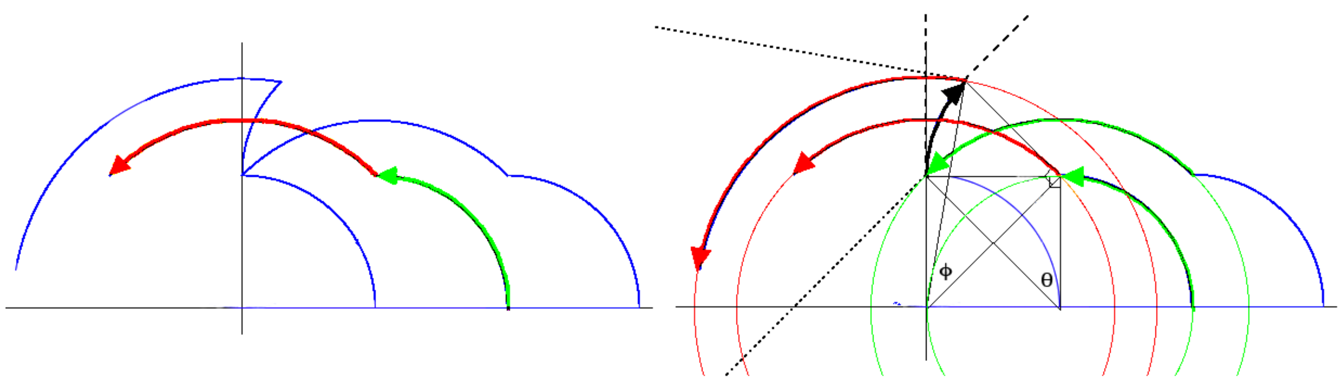

Figure 11: Acute corner produces two obtuse corners

Now we know that the next kind of corner that will present itself in the invariant path is an obtuse turn where the arcs have opposite orientation and the turn is in the direction of the curvature of the second arc. We will now examine this type of turn.

Lemma 5 *An obtuse corner that connects two circle arcs of opposite orientation, produces in the next iteration: (1) an obtuse angle that connects two arcs of opposite curvature and (2) an acute angle that connects two arcs of the same orientation and turns in the opposite direction of curvature.*

Proof We know that when the back wheel is tracing an oriented curve and hits a corner where it must turn onto an arc of the opposite curvature, that the bicycle will turn in the direction of the second curve. This is because the bicycle always turns in the direction that creates the *shortest* arc.

The proof is similar to that of *Lemma 4*. When the back wheel reaches the corner, it must turn in the direction opposite to its curvature which forces the front wheel to turn $\pi/2 + \theta$ radians to go backwards. Then the front wheel draws a unit circle arc in the direction of the second arc. Now the front wheel will need to turn slightly backwards, making a turn of $\pi/2 - \phi$ to line up with the appropriate tangent line and continue going in that direction of curvature. See *Figure 12* .

This process creates three new arcs. Two arcs of opposite curvature connected by a turn of more than $\pi/2$ making it obtuse, and then two arc of the same curvature connected by a turn of less than $\pi/2$ making it acute. □

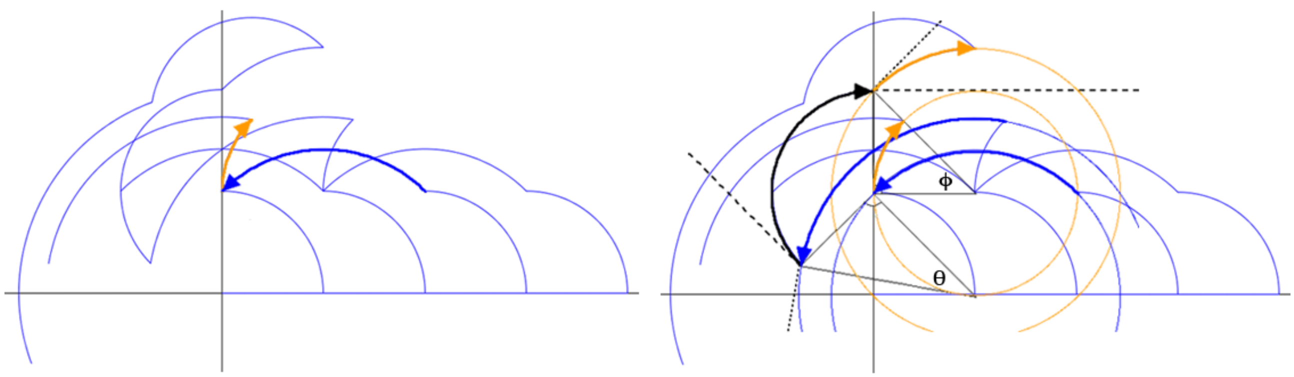

Figure 12: Obtuse corner produces an obtuse and an acute corner

As one can see, this second type of corner creates one corner of the second type and then another corner of the first type. Now we know that these two types of corners will never create another type of corner and will only produce each other as the iterations go to infinity.

We are ready to prove our next theorem about the growth of curvature of a discrete unicycle path.

Theorem 5 *The total absolute curvature of a non-trivial discrete unicycle paths grows exponentially through iterations.*

Proof We will examine only the invariant path of every discrete unicycle path, which we will suppose, without loss of generality, occurs in iteration 1. The discrete unicycle path is made up of straight lines, circle arcs and corners where straight lines and circle arcs connect. Of these components, the ones that contribute to total absolute curvature are arcs and corners.

When propagating from one iteration to the next, however, any circle arc has an image of a circle arc with the same total curvature. This is because the two arcs have the same center and the same angle. This means that the only elements that contribute to a change in total curvature from one iteration to the next are new corners that are created. New corners are the result of corners in the pre-image that turn into an arcs and must be connected to arcs on either side by corners.

We know from *Lemma 4* that an acute angle (which we will abbreviate by A) turns into two consecutive obtuse angles in the image. In addition, from *Lemma 5* we know that an obtuse angle in the invariant path (which we will abbreviate by O) turns into an obtuse angle followed by an acute angle in the image. Therefore, we can express this symbolically as $O \mapsto OA$ and $A \mapsto OO$.

Let A_n be the number of acute angles in the invariant path in the n^{th} iteration and likewise let O_n be the number of obtuse angles in the invariant path in the n^{th} iteration. Then it follows that for $n \geq 3$,

$$A_{n+1} = O_n \qquad \text{and} \qquad O_{n+1} = 2A_n + O_n \qquad \Longrightarrow \qquad 0 = O_{n+1} - O_n - 2O_{n-1}$$

We know $A_3 = 1$ and $O_3 = 0$. We can easily solve this linear recurrence equation, and find the equation for *all* the possible solutions is of the form:

$$O_n = \frac{4}{3}2^{n-3} + \frac{2}{3}(-1)^{n-3}$$

for $n \geq 3$.

This quantity grows exponentially which means that the number of obtuse angles in the invariant unicycle path grows exponentially. Therefore, even if the acute corners were set to 0 and the obtuse corners were set to $\pi/2$, the absolute sum of the corners would still grow exponentially through iterations. In addition, because the invariant path is in every unicycle path independent of the seed curve, this means that for every non-trivial discrete unicycle path the total absolute curvature grows exponentially. □

Another important fact about the complexity of discrete unicycle curves can be gathered from this theorem.

Corollary 5.1 *The total arc length of a non-trivial discrete unicycle paths grows exponentially in iterations.*

Proof *Theorem 5* proved that the number of obtuse corners in the invariant path grows exponentially. In the following iteration, each of these obtuse corners becomes a unit circle arc of angle greater than $\pi/2$. It follows that the sum of lengths of these arcs also grows exponentially through iterations, proving that the *total* length of each iteration must also grow exponentailly for every non-trivial discrete unicycle path.

More specifically, let θ_i be the angle measure of an arbitrary corner in the nth iteration of the invariant path and ϕ_i be the angle measure of an arbitrary circle arc in the same iteration with curvature equal to the inverse its radius $1/r_i$. Then from our definition of the total absolute curvature of iteration n,

$\kappa_n = \sum |\pi - \theta_i| + \sum |\frac{1}{r_i}\phi_i r_i|$. The first value in this sum becomes a circle arc in the next iteration and the second value becomes a concentric circle arc of longer length.

From this and *Lemma 1* we know that the total length of the $n+1$ iteration,

$L_{n+1} = \sum |\pi - \theta_i| + \sum |\frac{1}{r_i}\phi_i \sqrt{r_i^2 + 1}|$. Therefore,

$$L_{n+1} > \kappa_n$$

and since κ_n grows exponentially, L_{n+1} must grow exponentially as well.

□

An interesting corollary follows from the exponential grow of total length through iterations of the discrete unicycle path.

Corollary 5.2 *The expected number of times a line l intersects any non-trivial discrete unicycle path Γ increases exponentially as a function of the number of iterations completed.*

Proof As a result of *Corollary 5.1*, we know that the total length of a discrete unicycle curve Γ increases exponentially through iterations. Knowing this, we can use the Crofton Formula to examine the expected number of times a random line, l, intersects the path Γ [3].

Any directed line can be defined as a function of two parameters, it's distance from the origin, p, and its direction or angle away from the x-axis, φ. *Figure 13* displays two examples: $l_1 = (\varphi_1, p_1)$ and $l_2 = (\varphi_2, p_2)$. Let $n_\Gamma(l)$ be equal to the number of intersections that an arbitrary line l has with the curve Γ.

The Crofton's Formula states,

$$L_\Gamma = \frac{1}{4}\int\int n_\Gamma(\varphi, p)\ d\varphi\ dp.$$

We proved in *Corollary 5.1*, that the length of Γ grows exponentially through iterations. Therefore, the Crofton Formula implies that $n_\Gamma(l)$ is an exponential function as well. $\square$

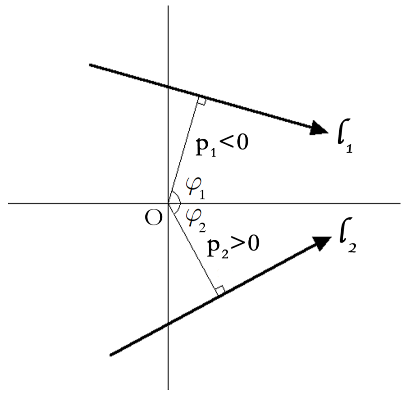

Figure 13: Directed lines with perpendicular distance from the origin p and angle from the x-axis φ

This means that the intersections of our curve with any line in the plane will increase exponentially through iterations. Recall, that in *Theorem 4* we proved that for any ray emerging from the corner x_{11} and any distance $d \in \mathbb{R}$, there is a point in the bicycle path that intersects the ray further away from the corner x_{11} than d. The Crofton Formula proves a stronger statement about the intersection of lines with the curve because it shows that not only must the path intersect with any line but the number of intersection points with a *random* line must also be exponential.

Theorem 5 and its corollaries are very important to the way we understand the growth of the discrete unicycle curves. We know from *Proposition 2* that at any moment, the unicycle path is contained in a circle whose radius grows *linearly.* We also know that the total absolute curvature and the length of the curve grow exponentially. This implies that the curve becomes very wrapped and twisted inside the disk as it grows.

Conjecture 1 We conjecture that any non-trivial discrete unicycle path is dense in $\mathbb{R}^2$.

We do have evidence to support the above conjecture by way of asking our computer program to draw many iterations of the invariant portion of all non-trivial discrete unicycle paths. We would provide a picture, but it would appear to readers like a black rectangle.

4 Smooth Model

In this section we will treat smooth unicycle paths, in which the seed curve and all resulting iterations are in $\mathcal{C}^\infty$.

Unlike the discrete model, in which a significant amount of the path is independent of the seed curve, as discussed in *Section 3.1*, a smooth bicycle path is entirely dependent on the seed curve.

This model requires new investigation tactics. We cannot directly apply the methods used for and Theorems proved about the discrete case to the smooth model. Approximating a smooth seed curve with a discrete seed curve proves problematic due to the different behavior of the bicycle in the discrete model when

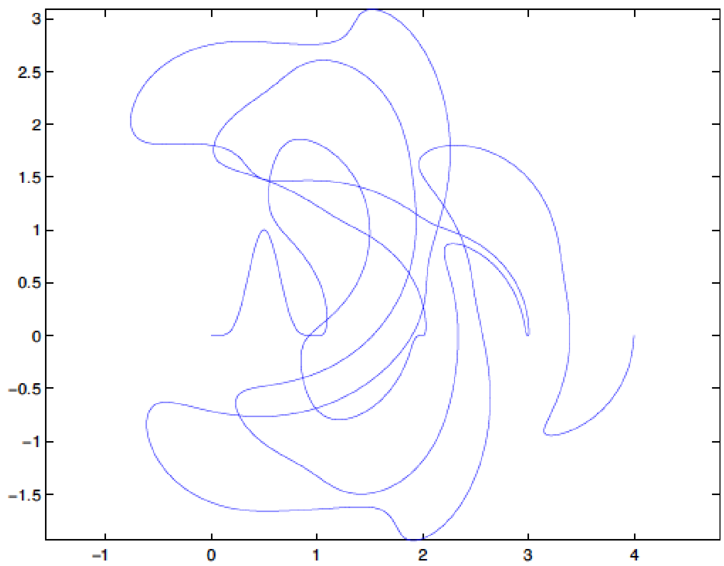

Figure 14: Smooth unicycle path [2]

the rear wheel arrives at a corner. This behavior causes arbitrarily small error to blow up after sufficiently many iterations. That is, we cannot stay arbitrarily close to the smooth curve we hope to approximate after many iterations.

However, note that the proof of *Proposition 2* does not require the seed curve to be discrete.

Recall *Propositions 1* in which Tabachnikov and Levi explore the number of extrema and the number of intersections with the x-axis of a smooth Γ, proving that these quantities grow at least linearly from iteration to iteration. We will continue this trend of exploring the growth of complexity of the curve by analyzing the points of inflection of the smooth unicycle path.

Proposition 6 *Denote by $P(\gamma_n)$ the number of points of inflection in the curve γ_n and assume that $P(\gamma_n)$ is finite. Then $P(\gamma_{n+1}) > P(\gamma_n)$ for any non-trivial smooth unicycle path.*

Proof We will distinguish vectors from scalars with bold font.

Suppose that $\boldsymbol{\gamma_n}$ is parameterized by arc length. The curvature of $\boldsymbol{\gamma_{n+1}} = \boldsymbol{\gamma_n} + \boldsymbol{\gamma_n'}$ is given by,

$$\kappa_{n+1} = \frac{\boldsymbol{\gamma'_{n+1}} \times \boldsymbol{\gamma''_{n+1}}}{|\boldsymbol{\gamma'_{n+1}}|^3} = \frac{(\boldsymbol{\gamma'_n} + \boldsymbol{\gamma''_n}) \times (\boldsymbol{\gamma''_n} + \boldsymbol{\gamma'''_n})}{|\boldsymbol{\gamma'_n} + \boldsymbol{\gamma''_n}|^3}.$$

But, using Frenet-Serret formulas, $\boldsymbol{\gamma''_n} = \kappa_n \mathbf{N} = \kappa_n \mathbf{J}\boldsymbol{\gamma'_n}$ where κ_n is the curvature of $\boldsymbol{\gamma_n}$ as a function of time, $\mathbf{N}$ is normal to $\boldsymbol{\gamma_n}$, and $\mathbf{J}$ is a rotation by $\pi/2$ in the plane. Thus,

$$\begin{aligned}
\boldsymbol{\gamma'''_n} &= (\kappa_n \mathbf{J}\boldsymbol{\gamma'_n})' \\
&= \kappa'_n \mathbf{J}\boldsymbol{\gamma'_n} + \kappa_n \mathbf{J}\boldsymbol{\gamma''_n} \\
&= \kappa'_n \mathbf{N} + \kappa_n^2 \mathbf{J}\mathbf{N} \\
&= \kappa'_n \mathbf{N} - \kappa_n^2 \boldsymbol{\gamma'_n}.
\end{aligned}$$

Additionally,

$$\begin{aligned} |\boldsymbol{\gamma'_{n+1}}| &= \sqrt{(\boldsymbol{\gamma'_n} + \boldsymbol{\gamma''_n}) \cdot (\boldsymbol{\gamma'_n} + \boldsymbol{\gamma''_n})} \\ &= \sqrt{1+\kappa_n^2}. \end{aligned}$$

Then we may calculate,

$$\begin{aligned} \kappa_{n+1} &= \frac{(\boldsymbol{\gamma'_n} + \boldsymbol{\gamma''_n}) \times (\boldsymbol{\gamma''_n} + \boldsymbol{\gamma'''_n})}{|\boldsymbol{\gamma'_n} + \boldsymbol{\gamma''_n}|^3} \\ &= \frac{(\boldsymbol{\gamma'_n} + \kappa_n \mathbf{N}) \times (\kappa_n \mathbf{N} + \kappa'_n \mathbf{N} - \kappa_n^2 \boldsymbol{\gamma'_n})}{(\sqrt{1+\kappa_n^2})^{3/2}} \\ &= \frac{\kappa_i + \kappa'_n + \kappa_n^3}{(1+\kappa_n^2)^{3/2}}. \end{aligned}$$

The denominator of this equation is always positive, so to find the zeros of κ_{n+1}, we simply consider the numerator.

Let F be an antiderivative of $1+\kappa_n^2$. Than one has,

$$\begin{aligned} \kappa_n + \kappa'_n + \kappa_n^3 &= e^{-F}(e^F \kappa_n)' \\ &= F'\kappa_n + \kappa'_n. \end{aligned}$$

Consider two consecutive zeros of κ_n occurring at $n \le t_1 \le n+1$ and $n \le t_2 \le n+1$. Any zero of κ_n will be a zero of $e^F \kappa_n$. Rolle's Theorem states that there is a zero of $(e^F\kappa_n)'$ in (t_1, t_2), and hence a zero of κ_{n+1}, occurring in (t_1, t_2).

Additionally, our construction requires that $\kappa_0^{(m)} = 0$ at $t = n$ and $t = n+1$ for all derivatives, and thus, that $\kappa_{n+1} = 0$, at $(n, 0)$ and $(n+1, 0)$.

Together, if κ_n has m zeros in (t_1, t_2), then κ_{n+1} must have at least $m-1$ zeros in (t_1, t_2) along with a zero on $[n, t_1)$ and a zero on $(t_2, n+1]$, that is, at least $m+1$ zeros altogether. Thus, we have $P(\gamma_{n+1}) > P(\gamma_n)$ for any non-trivial smooth bicycle path.

5 Smooth Closed Convex Shapes

In this section we will alter the construction of our path slightly. Instead of seed curves, we will use *seed ovals*.

Definition A *seed oval*, α_0, is convex, closed curve from which the rest of the path is constructed.

When the bicycle begins its motion, the front wheel moves so that the back wheel traces along this seed oval in a fixed direction, i.e. clockwise or counterclockwise around a central point. However, in this construction, the bicycle would be stuck in an infinite loop if we did not stop it or add a rule. Thus, we add a rule: when front wheel of the bicycle has completely constructed the resulting path, α_1, we pick up the bicycle and place the rear wheel on the α_1 such that the bicycle frame is tangent to the path pointing in the same direction as it had been traveling on α_0, i.e. clockwise or counterclockwise around the original central point. This process repeats to create infinite closed curves. In this way we can create infinite iterations as in previous sections, but here our bicycle path is necessarily discontinuous.

As in the previous construction, we can have a discrete model treating only circle arcs and line segments, but here we will only observe the smooth case in which $\alpha_0 \in C^\infty$.

If α_0 is a circle of radius r, then α_n is a concentric circle of radius $\sqrt{r^2+n}$ for all $n \in \mathbb{N}$ as described in *Lemma 1 (a)*. Thus, in this construction, α_0 results in infinite complete concentric circles. A perfect circle seed oval in this construction is analogous to a straight line segment seed curve in the previous construction.

Conjecture 1 If α_0 is not a circle then $\exists n \in \mathbb{N}$ such that α_n is not convex.

Figure 15, in which 24 iterations of an ellipse α_0 with semiaxes 1 and 1.01 evidences this conjecture.

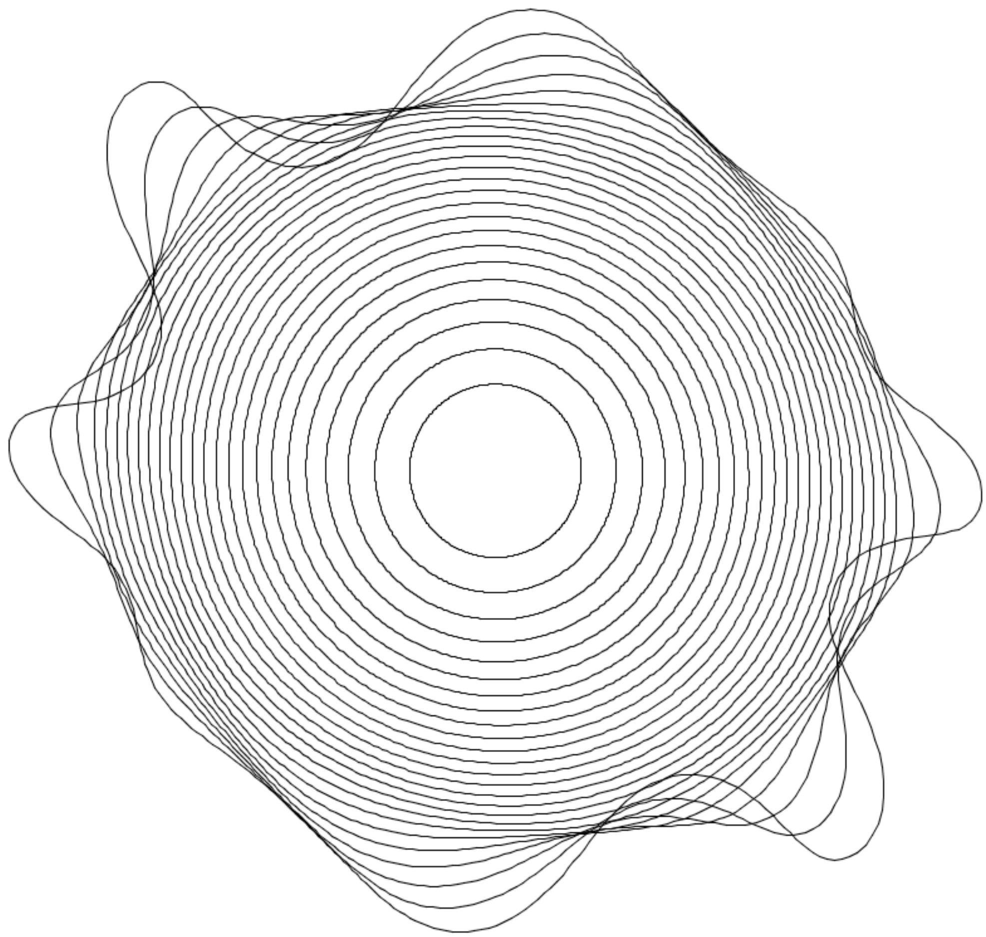

Figure 15: 24 iterations of an ellipse α_0 with semiaxes 1 and 1.01

Small perturbations from a circle become exaggerated through iterations. One can attempt to *normalize* this effect by alternating the direction in which the bicycle traverses between iterations, i.e. if the bicycle travels counterclockwise on α_n about a center point, then have the bicycle travels clockwise on α_{n+1} about the central point. But tweaking the construction in this way does not seem to prevent the bicycle path from becoming non-convex through iterations. What's more is tweaking the construction in this way does not even seem to slow down the appearance of points of non-convex iterations, though it does cause the path to *spiral* slower.

Figure 16, on the left, displays 11 iterations of an ellipse α_0 with semiaxes 1 and 1.1 with the regular construction, i.e. with the bicycle always traveling counterclockwise on the path. On the right, *Figure 16* displays 9 iterations the same α_0 with the attempted *normalizing* construction, i.e. with the bicycle alternating its direction of travel between iterations. In both constructions it appears to take approximately 8 iterations for the bicycle path to become non-convex.

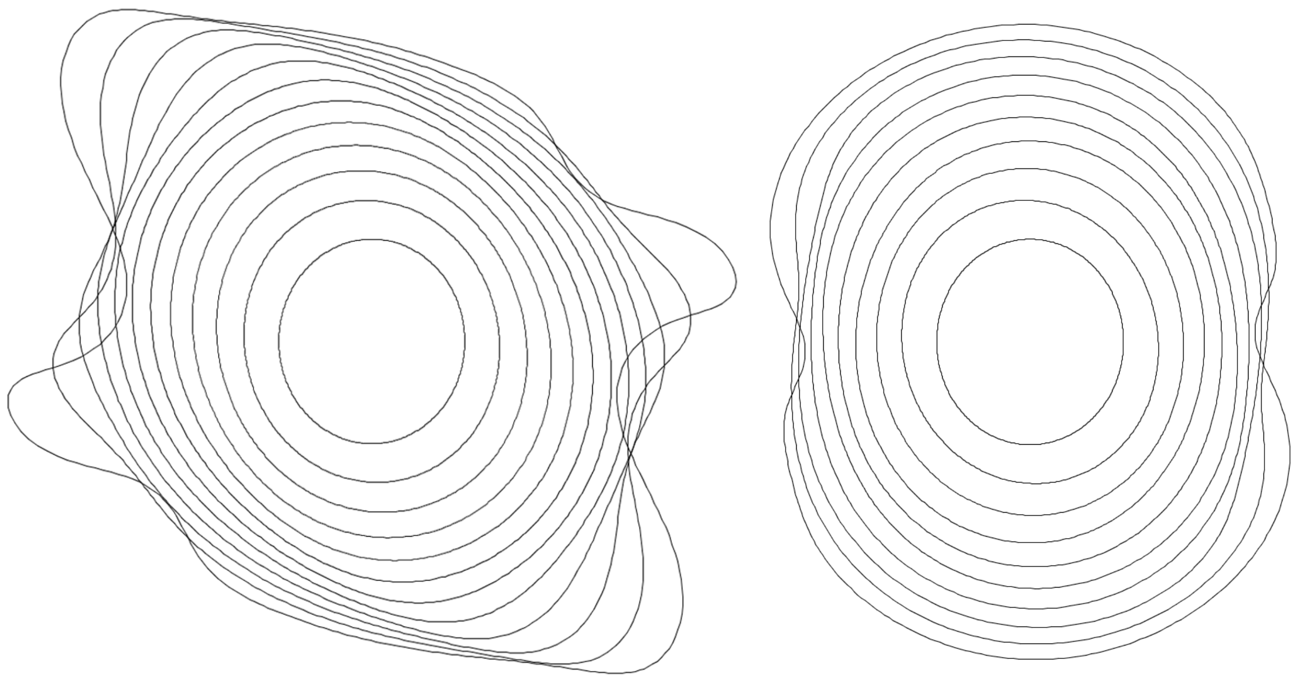

Figure 16: Ellipse α_0 with semiaxes 1 and 1.1

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