Converse Desargues' Theorem

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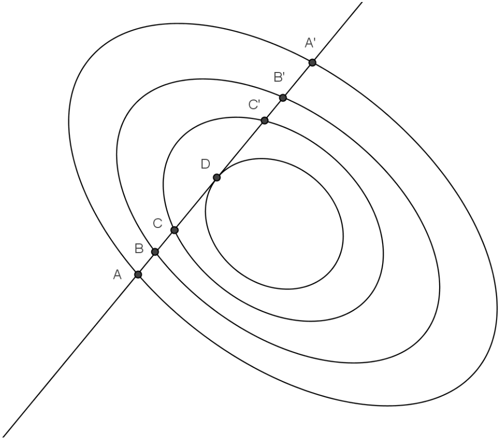
Theorem: (Desargues' Involution Theorem)

Consider a pencil of conics χ. If a line ℓ does not contain any of the base points of the pencil χ, then χ induces a projective involution on ℓ.

The *base points* of a pencil of conics are just the intersection points of the two conics defining the pencil.



A projective involution is a projective transformation of period two, given by the fractional linear transformation $t \mapsto \dfrac{t}{kt-1}$.

Converse Desargues' Theorem

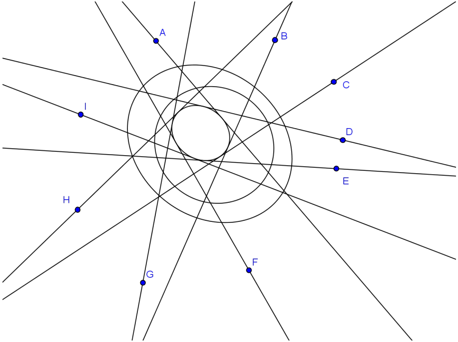
Let $f(x, y)$ be a (non-homogeneous) polynomial with a non-singular value 0. Let γ be an oval which is a component of the algebraic curve $f(x, y) = 0$.

Assume that the curves $\gamma_\epsilon = \{f(x, y) = \epsilon, \epsilon > 0\}$ foliate an outer neighbourhood of γ and that for each tangent line ℓ to γ its intersections with the curves γ_ϵ define a (local) projective involution on ℓ.

Conjecture : The curve γ is an ellipse and the curves γ_ϵ form a pencil of conics.

9-lines Theorem

Consider three nested ellipses and 9 lines tangent to the innermost one. If each of the lines intersects the other two ellipses in points which are pairs of an involution on the line, then the three ellipses must lie on a pencil.



Two conics and an algebraic curve

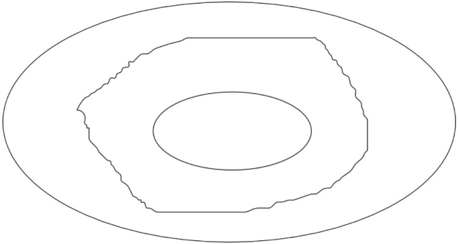
Consider three ovals nested inside each other, two of which are ellipses, with the property that any line tangent to the innermost one intersects the other two in pairs of points in involution.

Theorem

All three curves have to be ellipses.

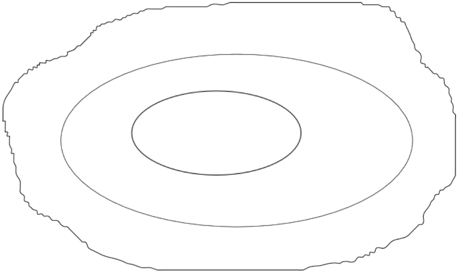
Case 1

The innermost and outermost ones are ellipses, and the one between them is any convex algebraic curve.



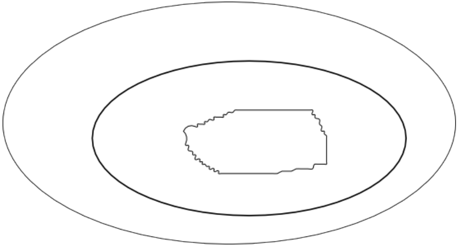
Case 2

The two inner curves are ellipses, while the outermost one is a convex algebraic curve.



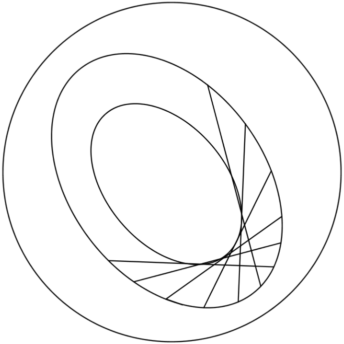
Case 3

The innermost curve is any convex algebraic curve, while the other two are ellipses.



We apply a projective transformation which takes the outer most ellipse into a circle, and the system then reduces to outer billiards in the hyperbolic plane, as studied by S. Tabachnikov

Area construction for hyperbolic billiards



Billiards in the Hilbert metric

Working in the real affine space V, which in our case is assumed to be finite dimensional, we let C be a non-empty bounded convex open subset of V.

Distance the Hilbert metric is hence defined as follows: Let $x, y \in C$. Denote by $h(x, y)$ the distance between x and y in the Hilbert metric. In the case where $x = y$, we naturally set $h(x, y) = 0$. If $x \neq y$ then the well defined line containing x and y cuts the boundary of C in two points say a and b. Now, similar to the hyperbolic metric, we set

$$h(x, y) = \log[x, y, a, b] = \log\left(\frac{a-x}{a-y} : \frac{b-x}{b-y}\right)$$

Hilbert metric in an equilateral triangle.

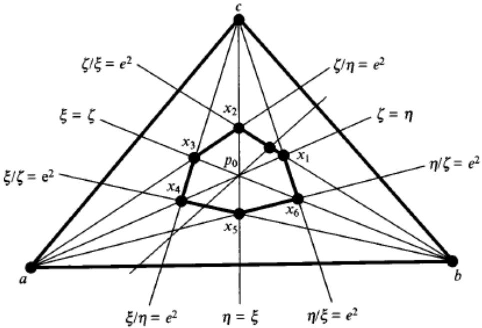


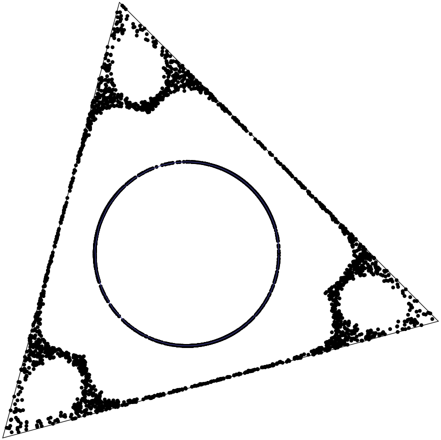
Figure 5: A sphere (!).

Let C be the interior of a convex polygon $\overline{C}$ witn N vertices in a 2-dimensional plane V.

- If $N \geq 4$ the groups $Coll(C)$ and $Isom(C)$ are finite.
- If $N = 3$ the group $Coll(C)$ is isomorphic to the natural semi-direct product $\mathbb{R} \rtimes D_3$.

  In particular the group $Isom(C)$ acts transitively on C iff C is a triangle.[de la Harpe, 2001]

Plot of iterations of outer billiard map in the Hilbert metric



Conjectures and open problems

- Other possible paths to explore include relaxing the constraint about the lines being tangent to some given innermost curve and just considering a general two-parameter family of lines traversing through the conics/polygons in this involutive fashion.
- Lastly, we can generalize in terms of dimension and consider outer billiards in ellipsoids.