Outer Billiards with Contraction

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Presented on August 6^{th}
(modified on August 9^{th})

Summary

- In section I, we discuss stability issues, shedding light on a striking difference between the regular pentagon and the regular septagon.

Summary

- In section II, we discuss trapezoidal outer billiards with contraction. We can prove that for any trapezoid there are infinitely many stable degenerate periodic orbits (SDPOs).

Summary

- In section III, we outline a proof that for certain choice of the polygon and the contraction, attracting Cantor sets exist.

A Comparison

	Square / Triangle	**Trapezoid**	**Reg. Pentagon**	**Reg. Septagon**
Finiteness in bounded ball	Yes	Yes (?)	No	No (?)
Exotic Periodic Orbits	None	Infinitely many	None (?)	???
Stability of Periodic Domains	Yes	Yes (?)	Yes	No !
Possible Approach	Hierarchy of the tiling	Horizontal Slicing / quasihierarchy	Renor-malization / quasihierarchy	???????

I. Stability

- **Theorem. (Stability Criterion)**
- Given a periodic domain Q for P, look at reflected images of P determined by the combinatorics of Q. Then Q is $\lambda-$stable if and only if the barycenter of all the images of P lies in the interior of Q.

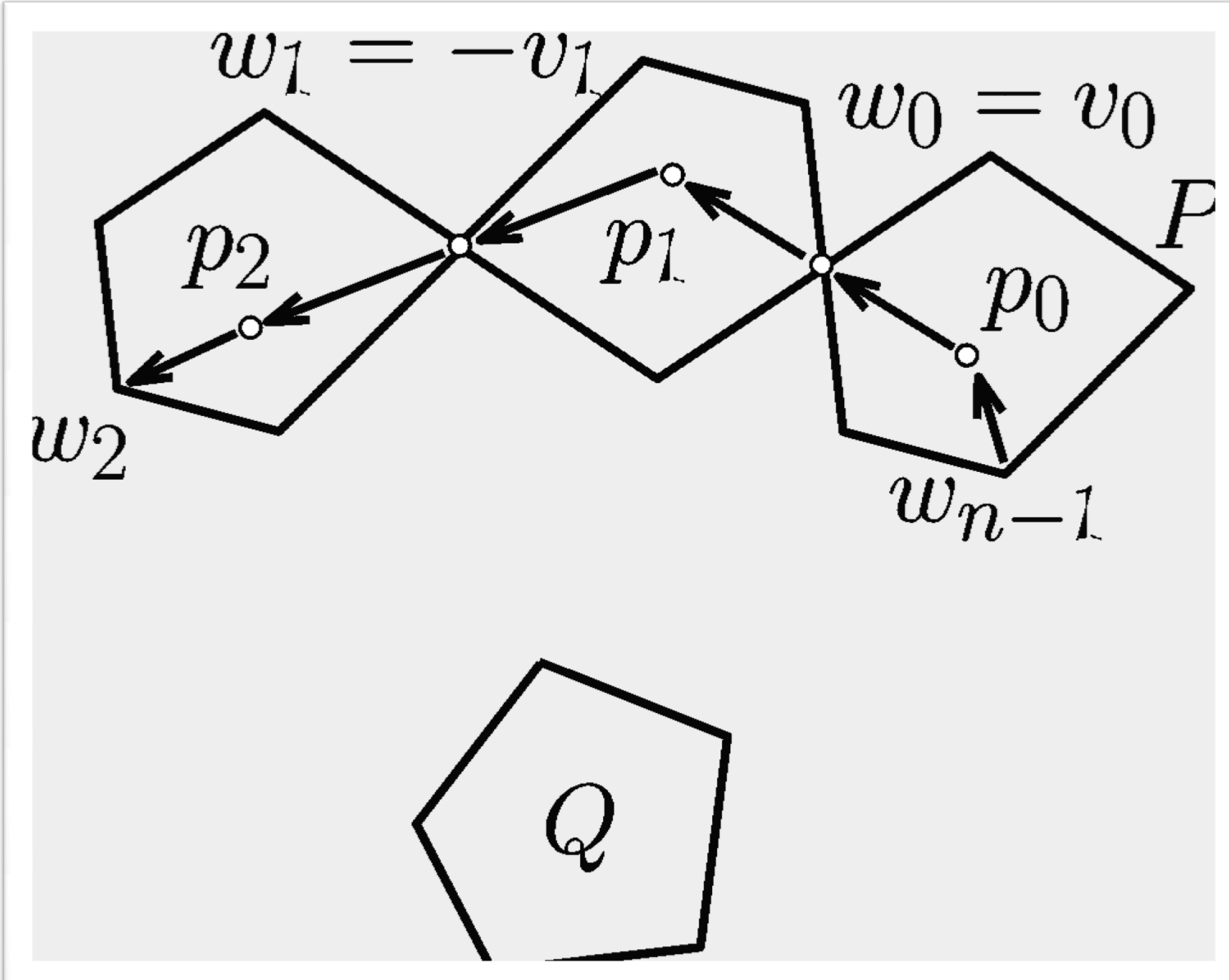
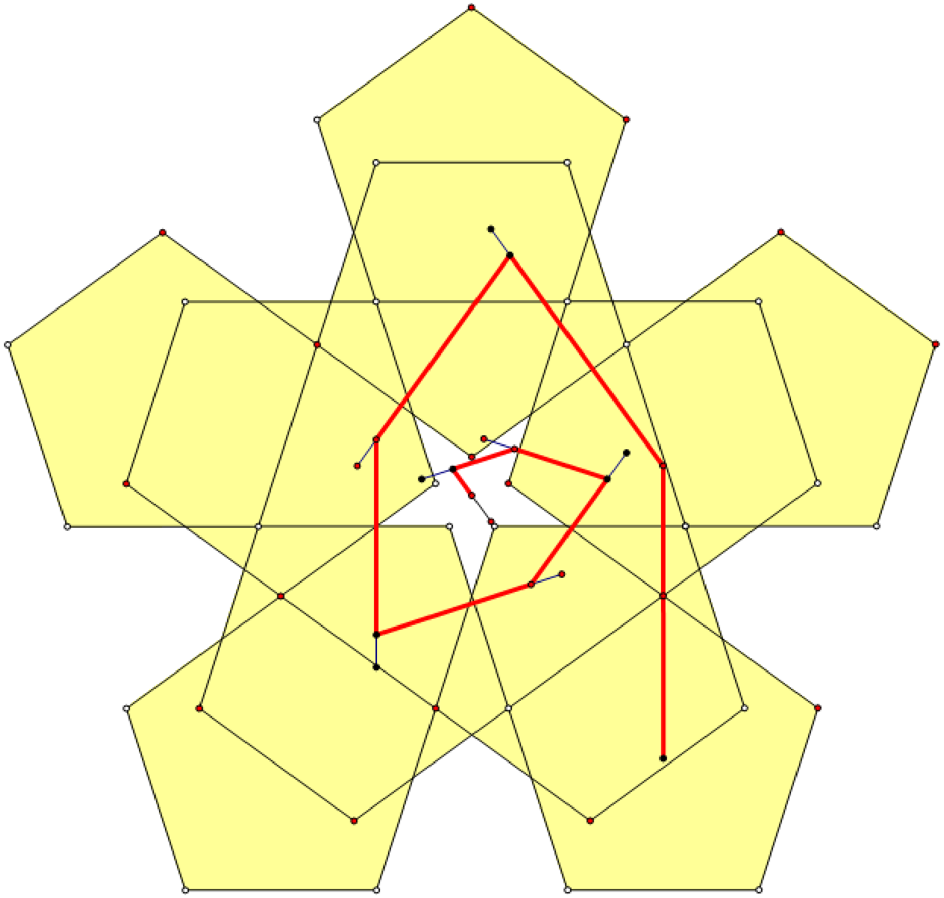
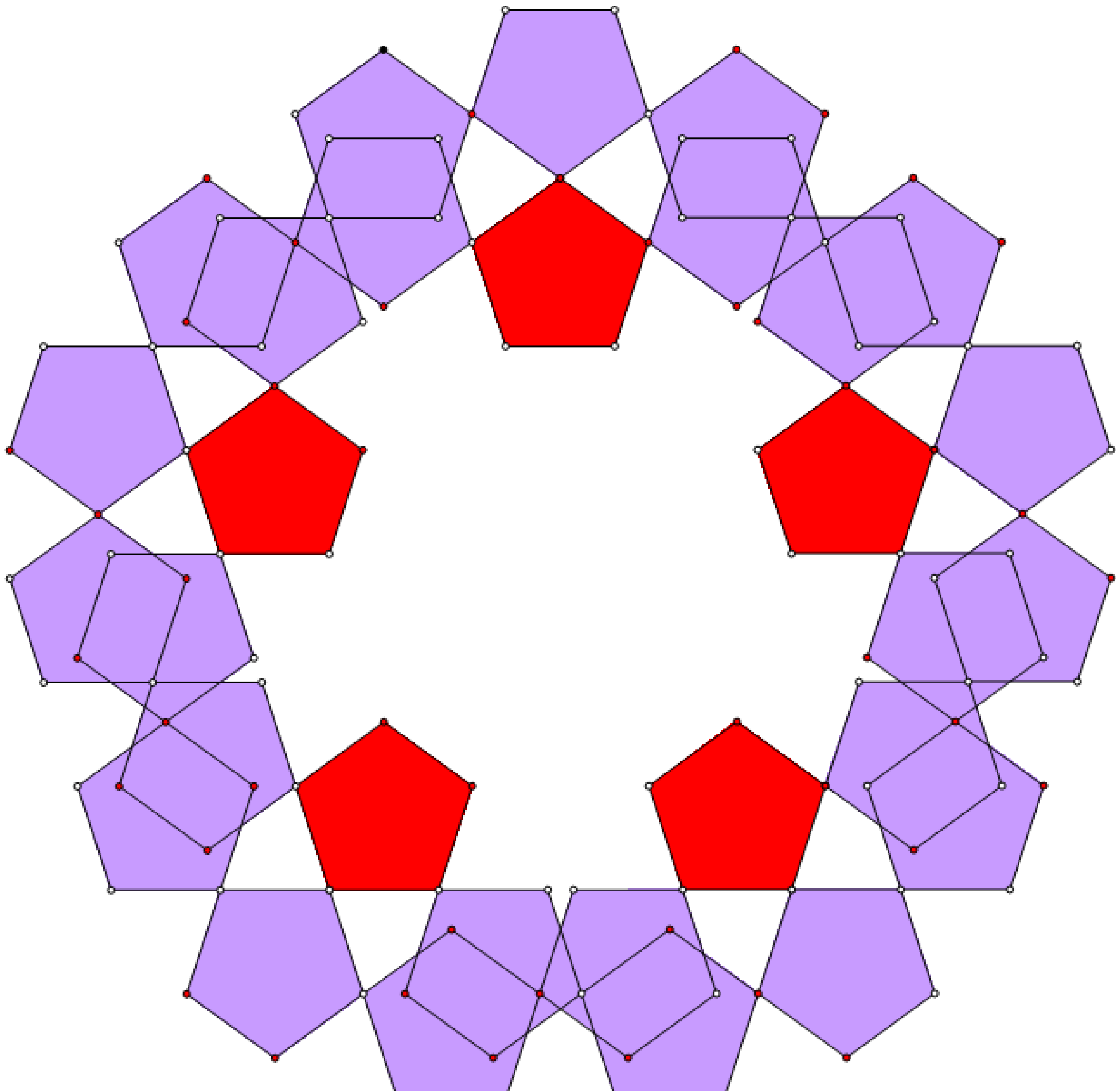

$w_1 = -v_1$
$w_0 = v_0$
P
p_2
p_1
p_0
w_2
w_{n-1}
Q

Illustration of the Stability

Period 10 orbit



Period 30 orbit



- **Corollary**. *Symmetric* periodic domains are stable

- **Corollary**. *Odd* periodic domains are stable

- **Corollary**. When n = 3, 4, 5, 6, 8, *all* periodic domains for the regular n-gon are stable

- **Corollary**. If P is centrally symmetric, all Culter periodic domains are stable

Stability of the Regular Pentagonal Periodic Domains

Let (x, y, i) be the coordinates of the vertex ϑ as γ first enters an angle, and let (x_1, y_1, i_1) be its coordinates when γ first enters the next angle. Define the map

$$\phi \colon (x, y, i) \mapsto (x_1, y_1, i_1).$$

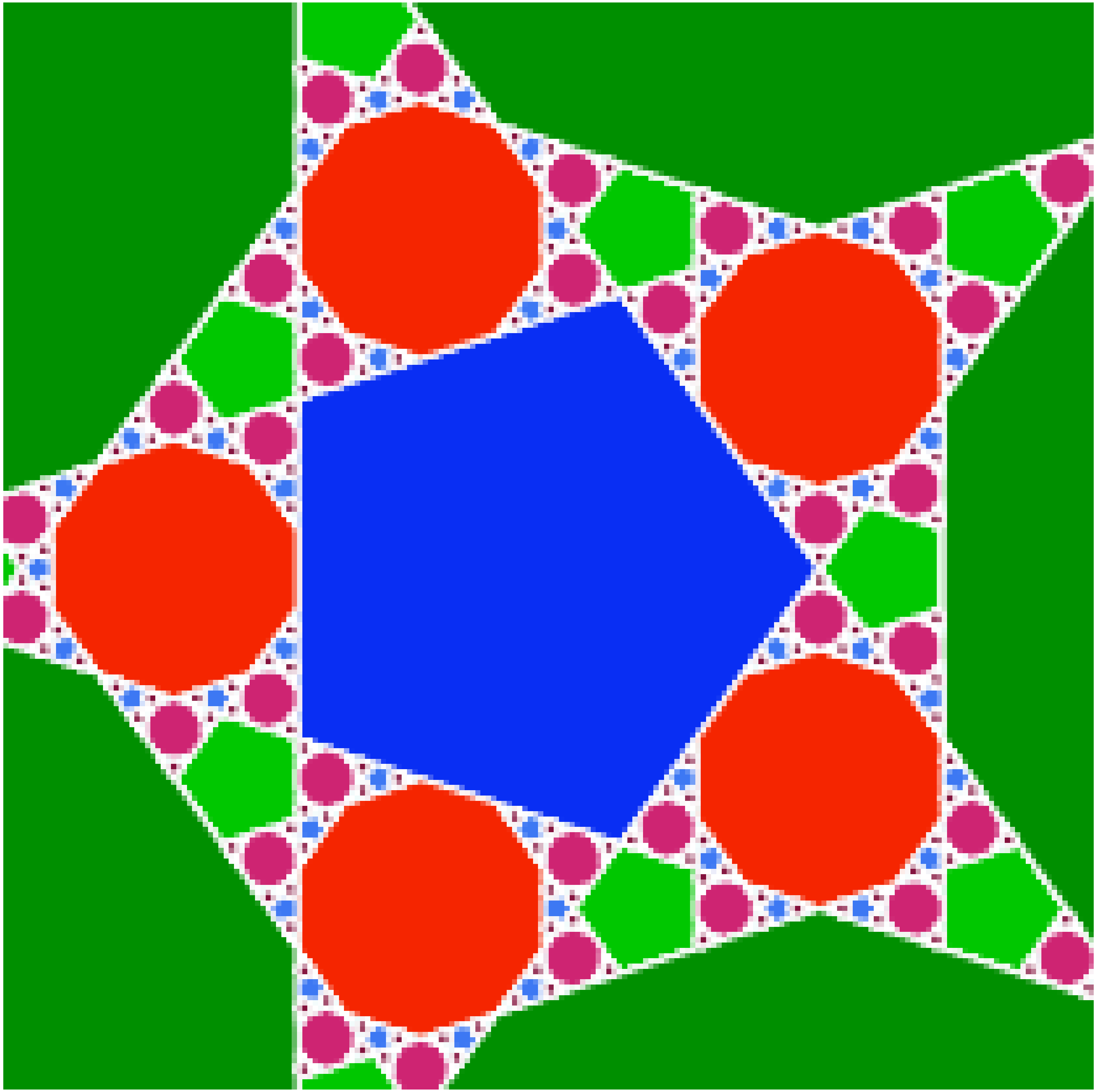
Let the side of γ be equal to 1; define the shift S (in each angle) by

$$S \colon (x, y, i) \mapsto (x + 3 + \sqrt{5}, y, i).$$

LEMMA. $S \ \phi = \phi \ S$.

The proof is straightforward.

S. Tabachnikov *Adv in Math* (1995)



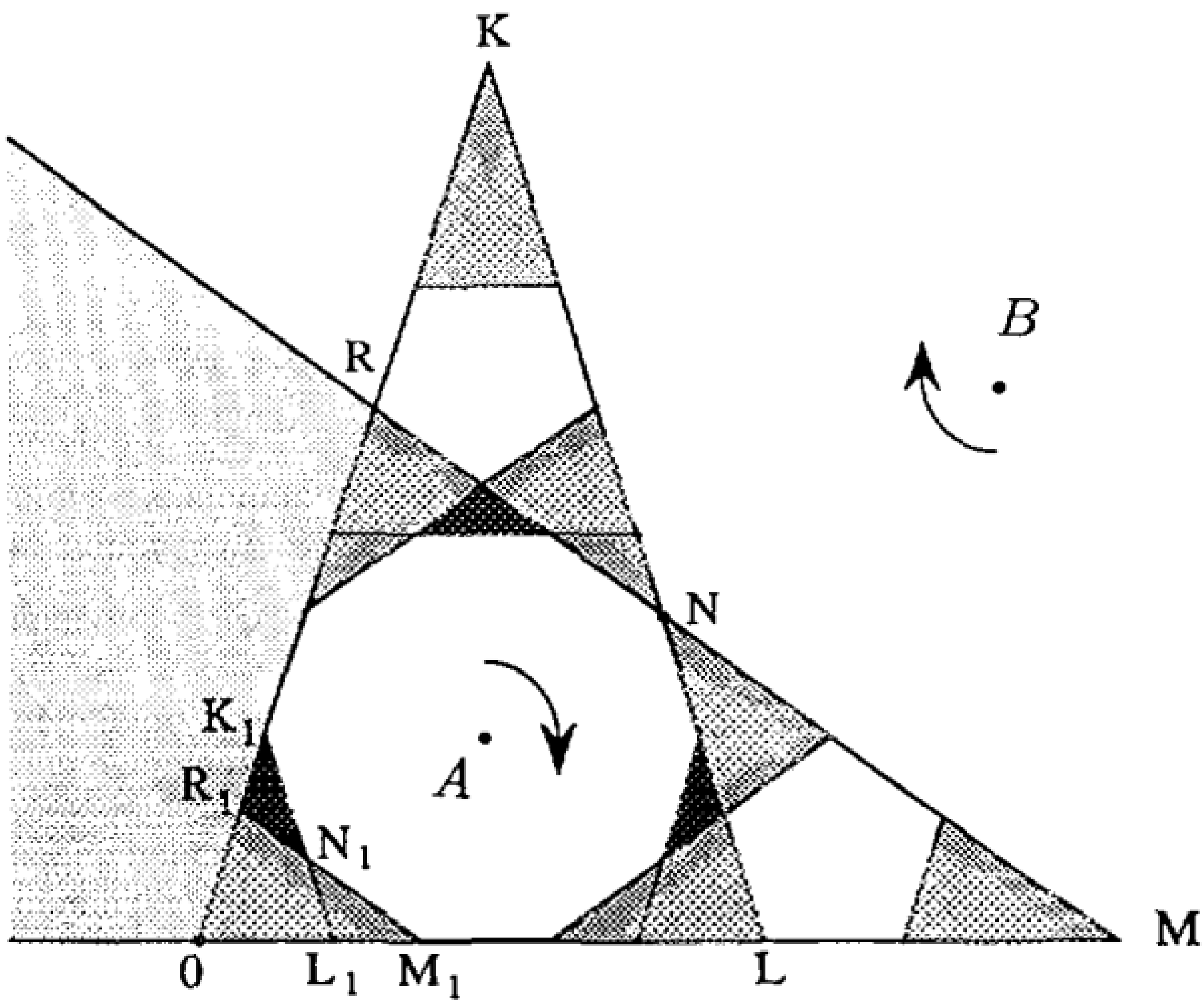
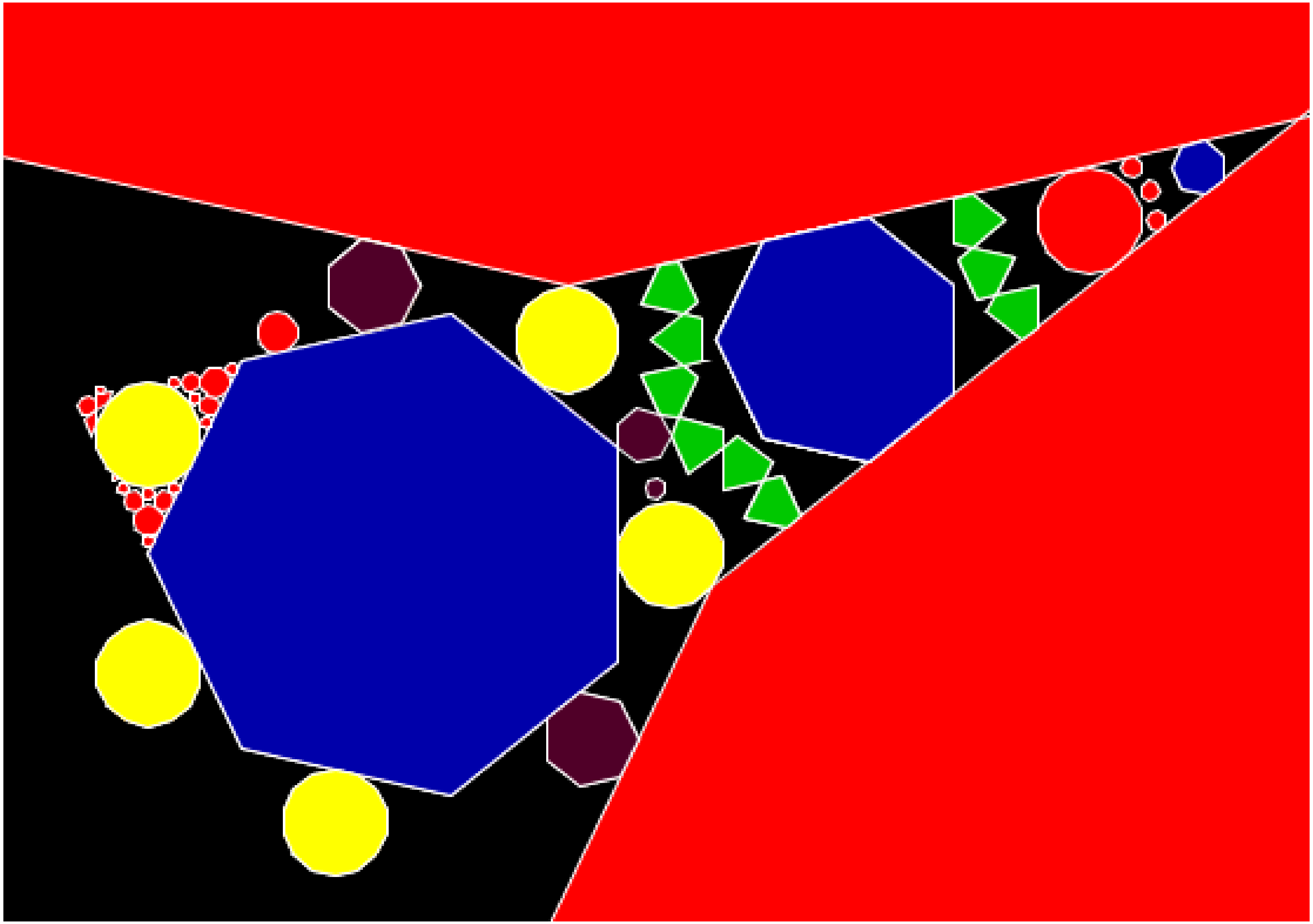


FIGURE 17

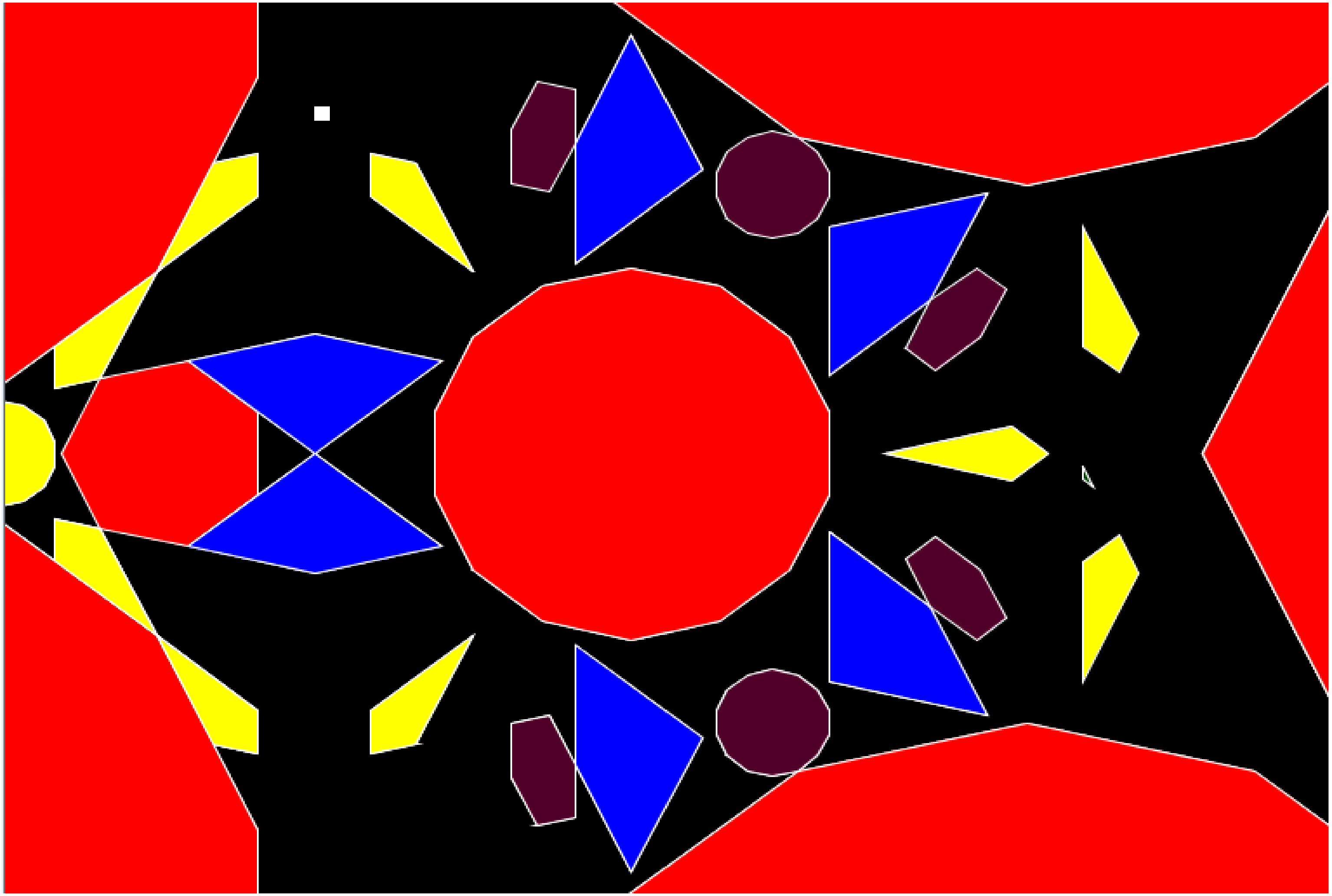
S. Tabachnikov *Adv in Math* (1995)

Regular Septagon: Unstable Pentagon



- Period 57848, Radius around 0.0001

Schwartz's Zoo of Exotic Domains



- Diameter: 0.002, Period: up to 500000

II. Trapzoids

3-5-7 Conjecture.

- *All the exotic periodic orbits* have periods ending with either 3, 5, or 7.

- Finiteness follows.

3-5-7 Weaker **Conjecture.**

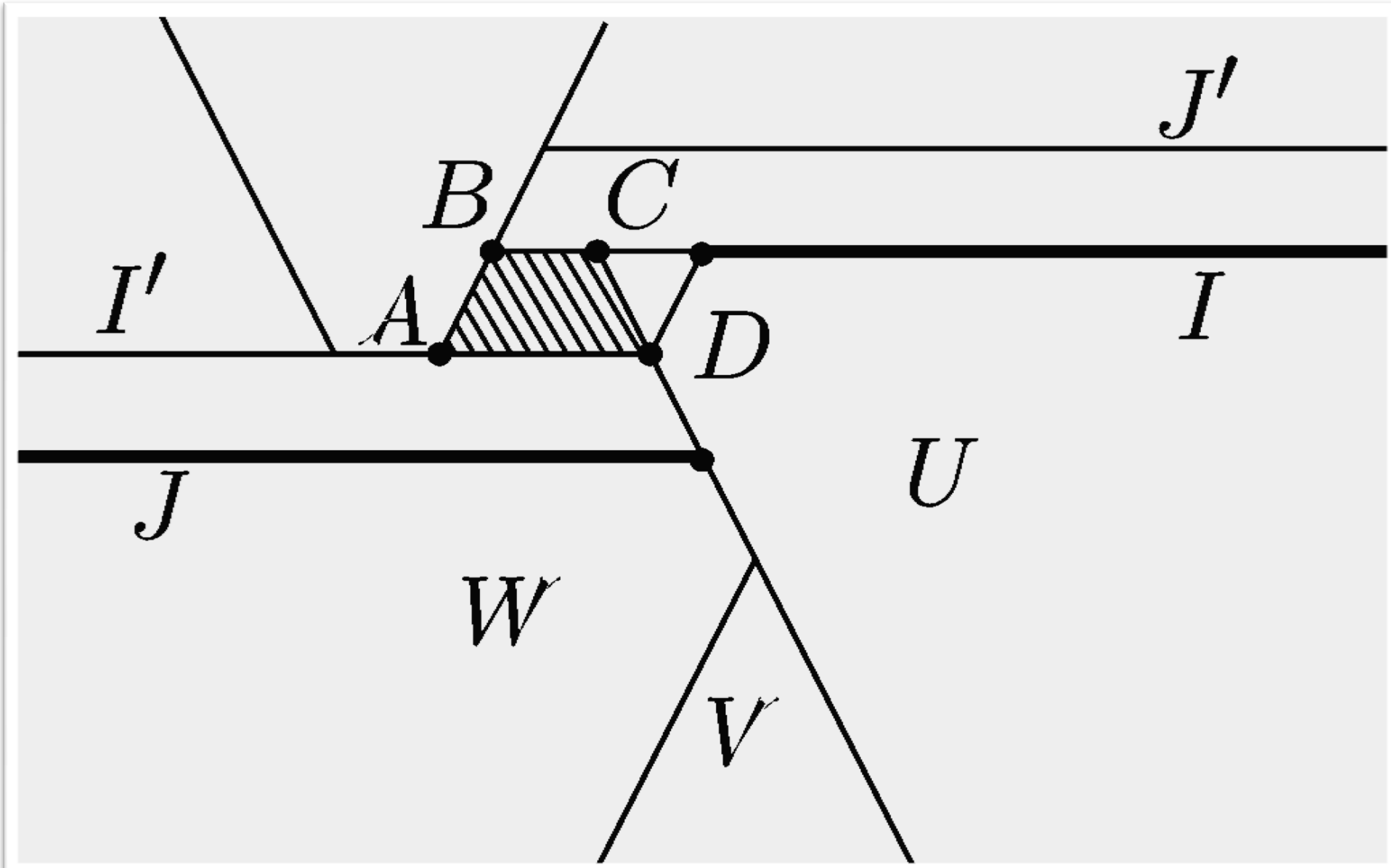
- *For any positive integer congruent to either 3, 5, or 7, there exists a* SDPO *with that integer as its period.*

- Infinitely many exotic periodic orbits!

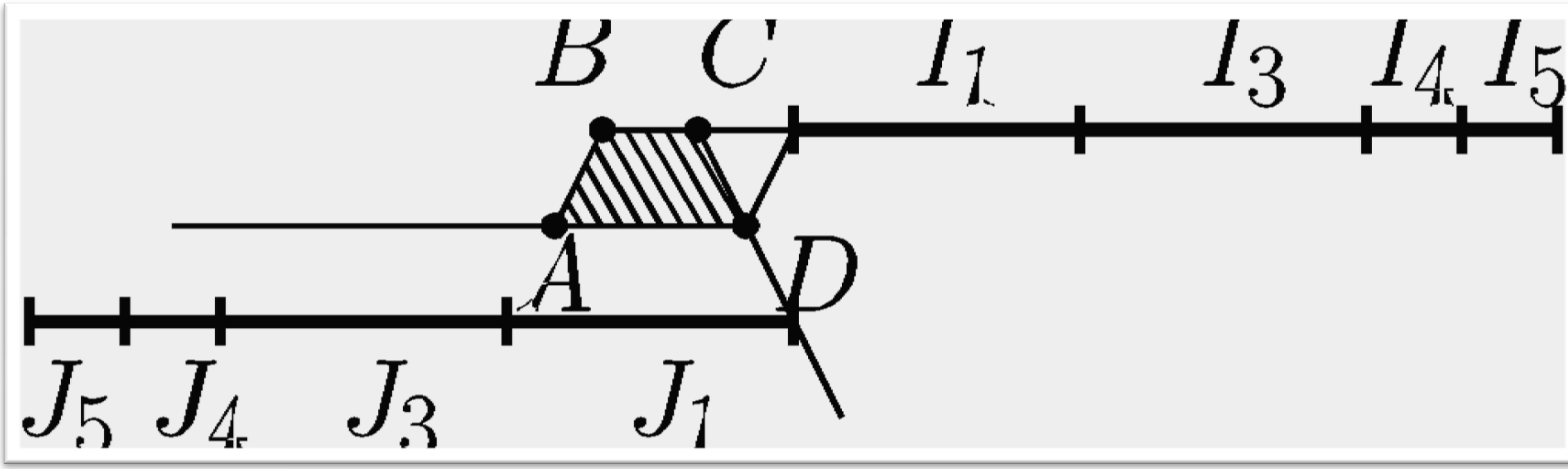
3-5-7 Weaker **Conjecture.**

- Weaker conjecture is proved.
- Boils down to studying **fixed points** of orientation-reversing interval exchange transforms

The "Pinwheel" Dynamics

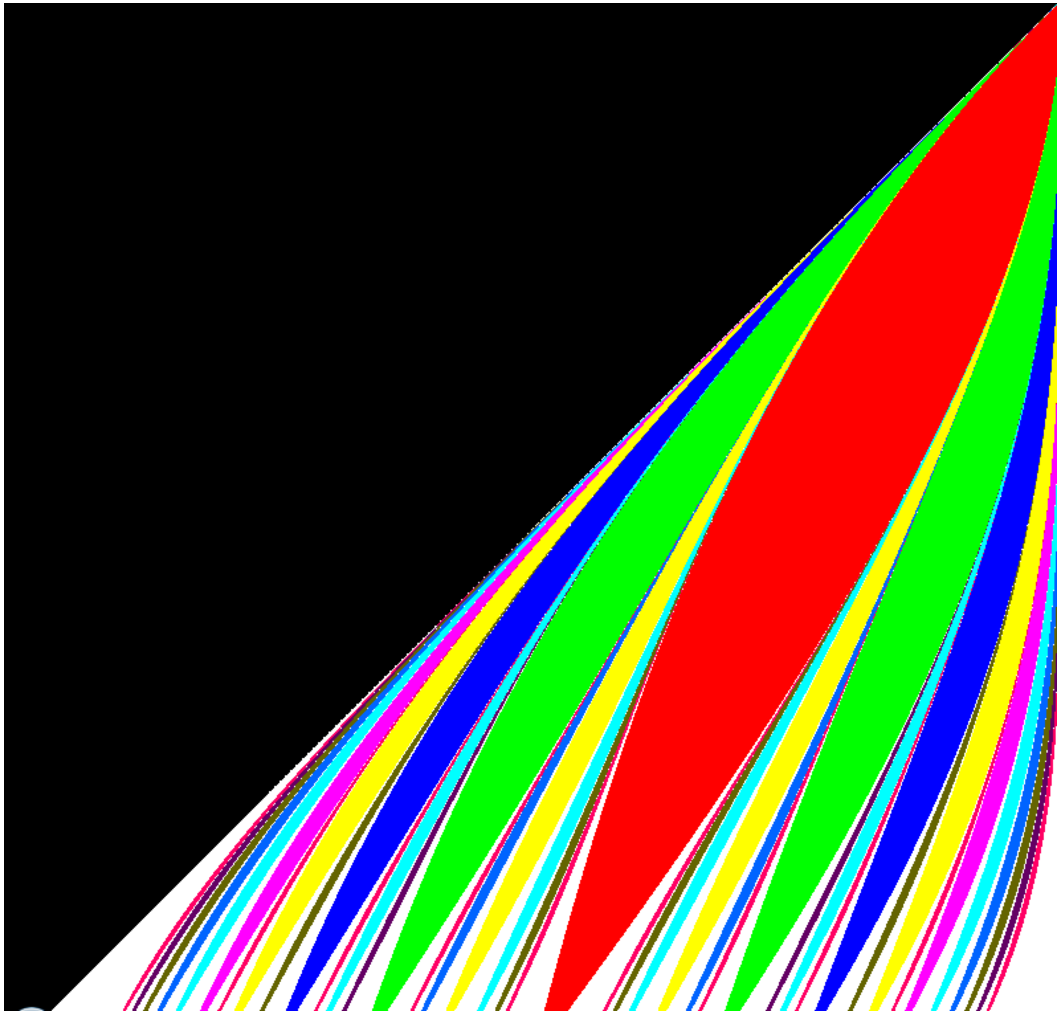


Interval Exchange Transform



Ⅲ. Cantor Set

Affine Contractions



- H. Bruin, J Deane *Proceedings of the AMS* (2008)
- Coloring Scheme by P. Hooper

Extending the Rotation Theory

- F. Rhodes, C. Thompson

Rotation number for monotone functions on the circle (1985)

Topologies and rotation numbers for families of monotone functions on the circle (1989)

- Summary
  - ***Well-defined and nice*** with respect to strictly increasing monotone maps of the circle
  - ***Continuous*** with respect to the Hausdorff metric on graphs

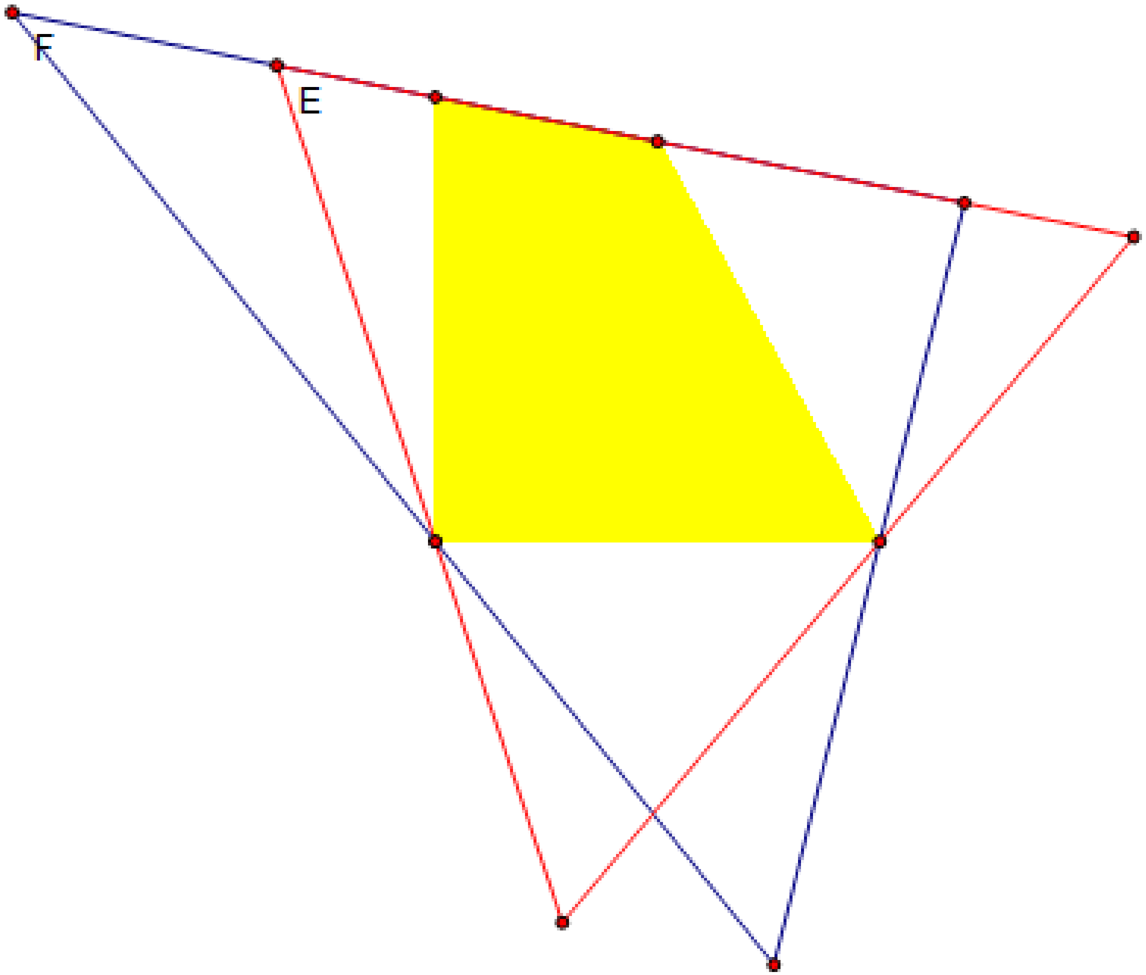
Extending the Rotation Theory (2)

- R. Brette

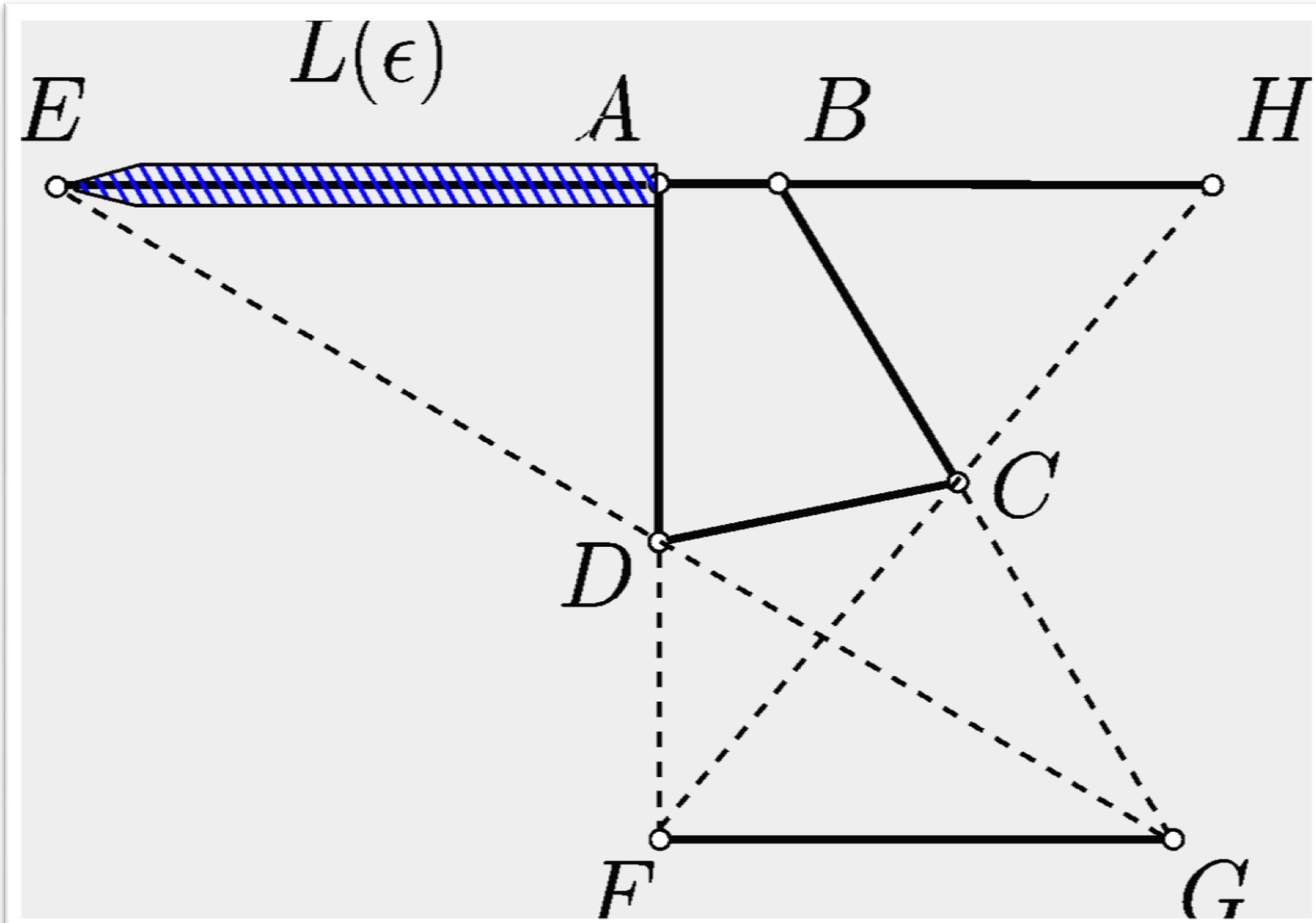
Rotation numbers of discontinuous orientation-preserving circle maps (2003)

- Summary
  - ***If rational,*** *all orbits are asymptotically periodic.*
  - ***If irrational****, the limit set is either the whole circle or a unique Cantor set*

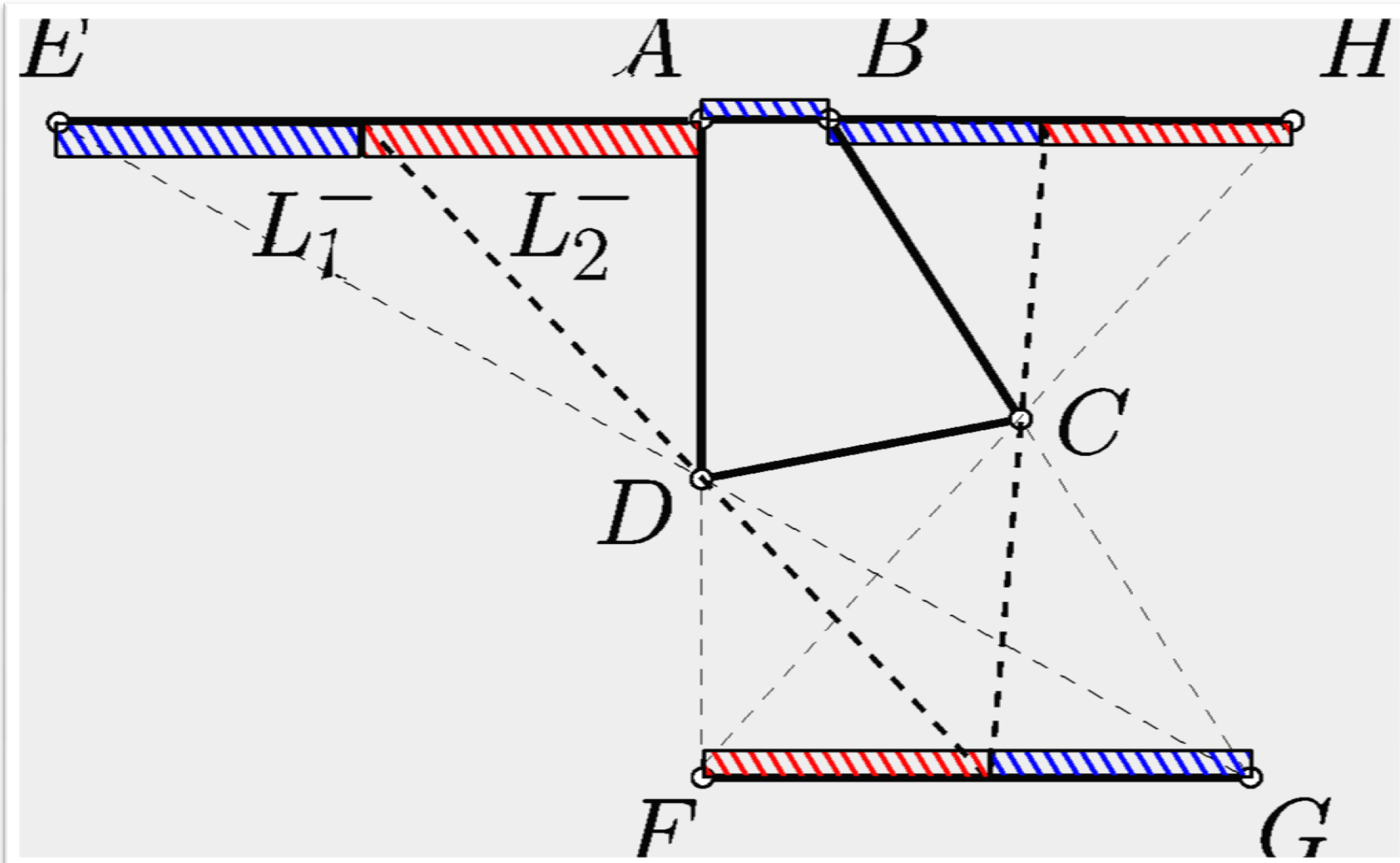
Triangular Transition on Quadrilaterals

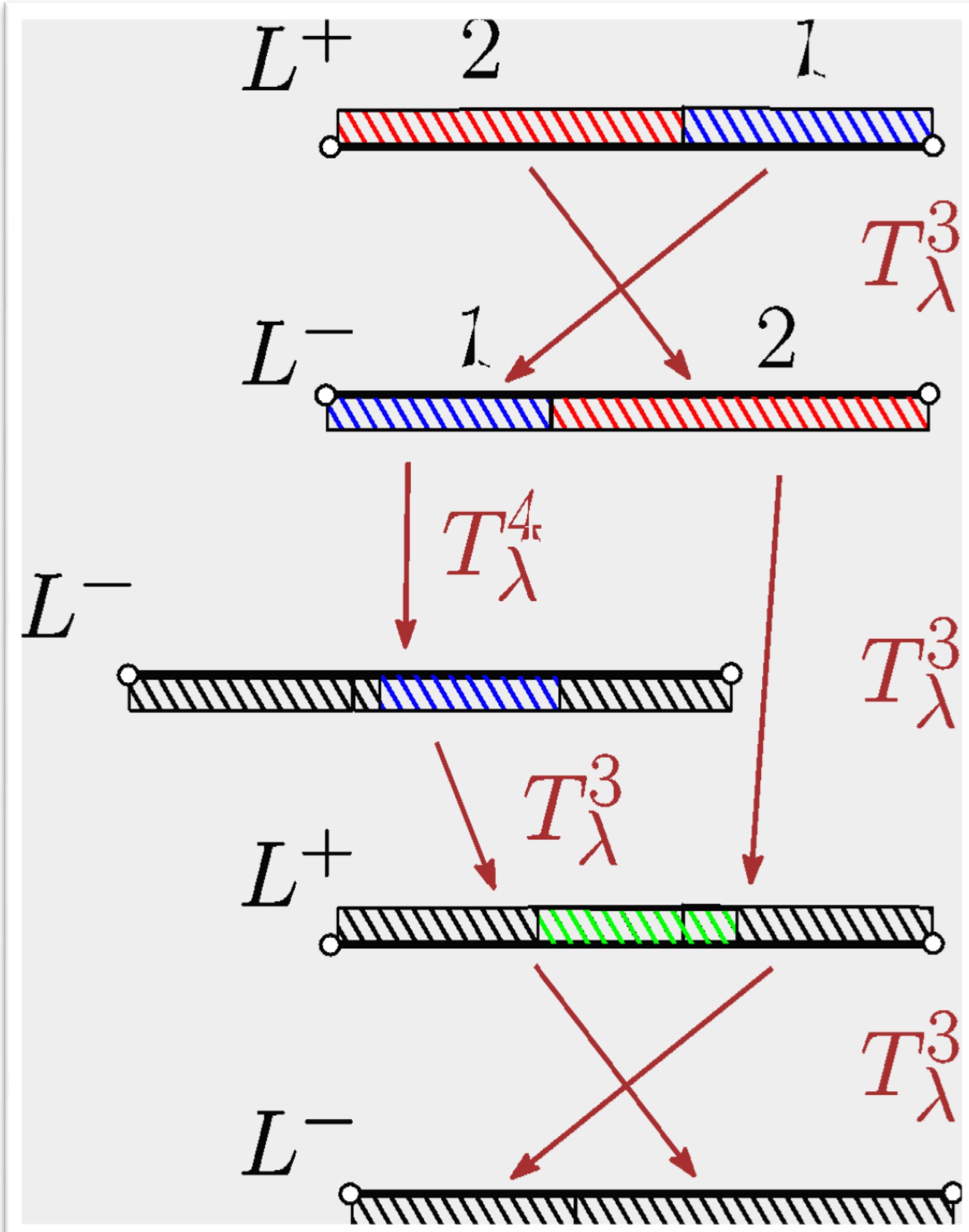


An Invariant Region



Dynamics of the Return Map



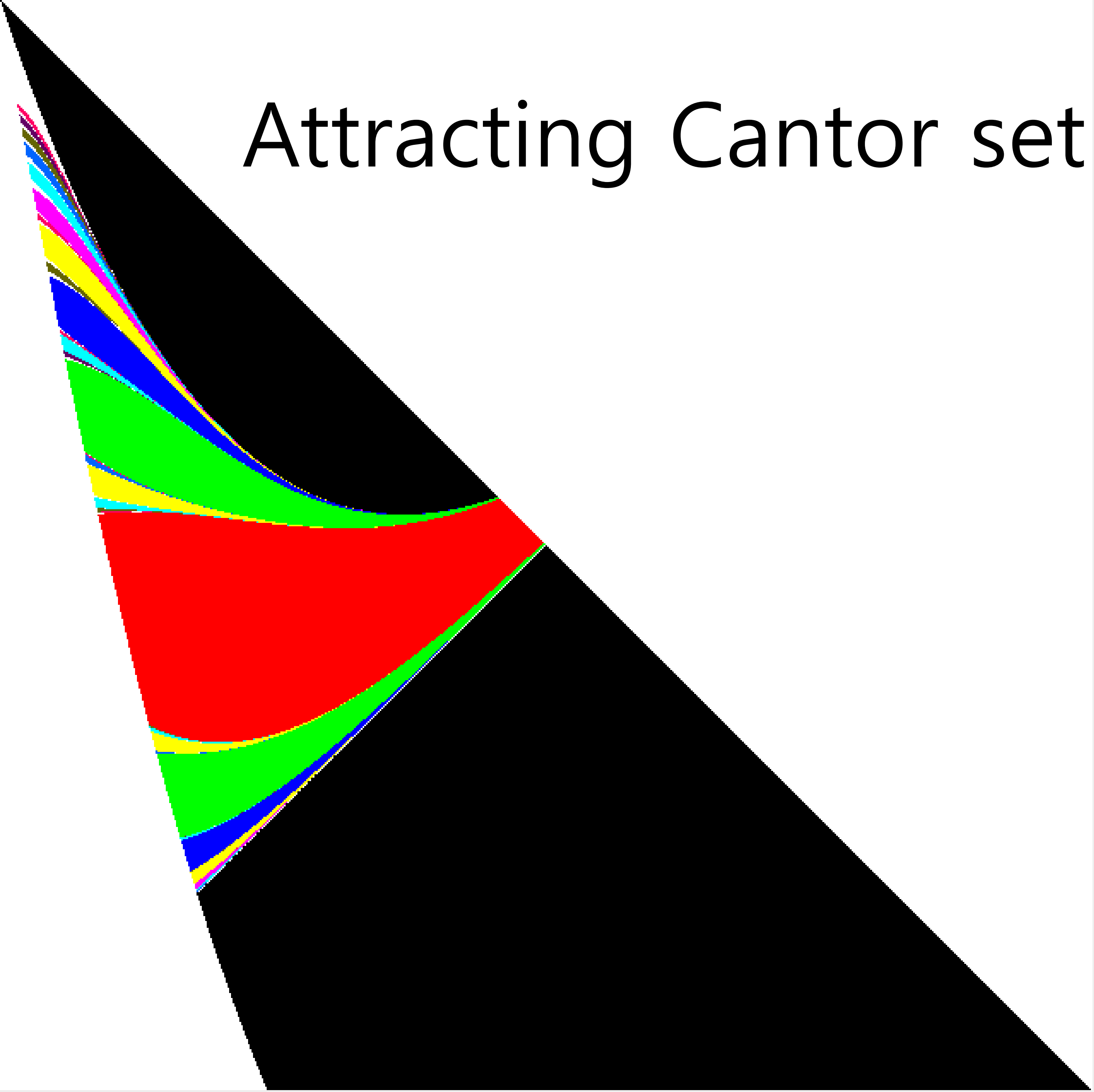


- Semiconjugacy

Proving existence of attracting Cantor set: **Ingredients**

- Rotation theory
- Bijection between periodic orbits
- Continuity of the rotation number

Attracting Cantor set Exists!



- We sincerely thank our advisors Prof. P. Hooper, Prof. S. Tabachnikov, Diana and Tarik.
- We also thank Prof. R. Schwartz. Many figures were generated by his program.
- We also thank all ICERM staff.