SUMMER@ICERM 2012

Research Projects

The document contains a list of possible research problems for the 2012 SUMMER@ICERM undergraduate research program.

Contents

1 Billiards and related systems

1.1 Polygonal outer billiards in the hyperbolic plane

The outer billiard about a convex polygon P in the plane $\mathbf{R}^2$ is a piece-wise isometry, T, of the exterior of P defined as follows: given a point x outside of P, find the support line to P through x having P on the left, and define $T(x)$ to be the reflection of x in the support vertex. See [22, 9].

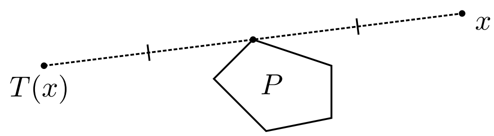

Figure 1: The outer billiard map in the plane

C. Culter proved (Penn State REU 2004) that every polygon in the plane admits periodic outer billiard orbits, see [24]. Outer billiard can be defined on the sphere and in the hyperbolic plane. On the sphere, there exist polygons without periodic outer billiard orbits. **Conjecture**: *every polygonal outer billiard in the hyperbolic plane has periodic orbits.* These orbits may lie on the circle at infinity.

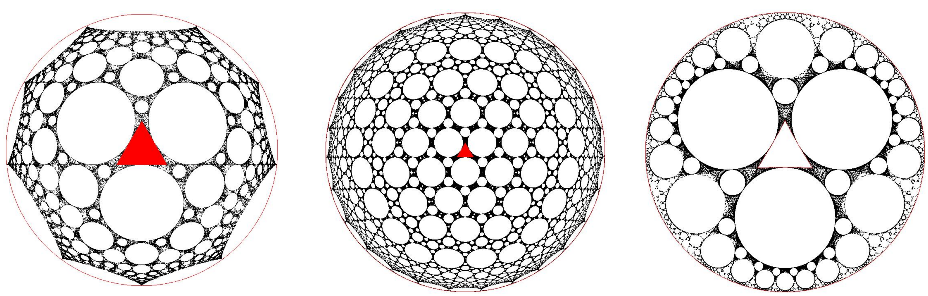

Figure 2: Outer billiards on equilateral triangles in the hyperbolic plane

Figure 2 illustrates the complexity of this system, even in the case of an equilateral triangle: the white discs are periodic domains of the outer billiard map.

Another interesting problem is to describe polygonal outer billiard tables in the hyperbolic plane for which all orbits are periodic. For example, right-angled regular n-gons (with $n \geq 5$) have this property, see [8]. In the affine plane, every outer billiard orbit about a lattice polygon is periodic.

1.2 Polyhedral outer billiards in 4-dimensional space

Let M be a closed convex hypersurface in $\mathbf{C}^2$. For a point $x \in M$, let $n(x)$ be the outer unit normal vector. The *outer billiard map* T of the exterior of M is defined as follows. For $t > 0$, consider points $y = x + itn(x)$ and $z = x - itn(x)$ (of course, $i = \sqrt{-1}$); then $T(y) = z$. One can prove that for every y outside of M there exists a unique $x \in M$ and $t > 0$ such that $y = x + itn(x)$, hence the map T is well defined. See [22, 9] for more details.

Problem: *study the dynamics of the outer billiard map when M is the surface of a regular polyhedron in $\mathbf{C}^2$.*

In dimension four, there are six regular polyhedra. Even for a regular simplex, one expects an interesting dynamical system.

In the plane, a regular pentagon (and other regular n-gons with $n \neq 3, 4, 6$) yields a beautiful fractal set, the closure of an infinite orbit of the outer billiard map. See Figure 3.

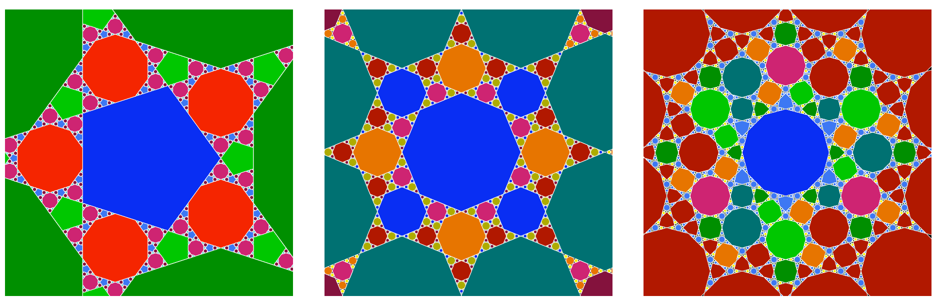

Figure 3: Outer billiards on regular n-gons: $n = 5, 8, 12$

1.3 Outer billiards with contraction

Fix a positive constant $\lambda < 1$ and consider a modified outer billiard where the reflection in support point is composed with the dilation with coefficient λ centered at this point. This transformation contracts the area with coefficient λ^2. The problem is to study the dynamics of this class of outer billiards. Here is sampler of problems.

Let the outer billiard table be a convex polygon. Prove that, for every λ, every orbit converges to a periodic orbit. What happens in the limit $\lambda \to 1$?

Let the outer billiard table be an oval. Describe the limit set of the orbits. Is it true that for every oval, except an ellipse, and for every λ (or for λ close *enough to 1), there exist periodic orbits?*

An interesting class of curves to study as outer billiard tables are piece-wise circular curves. For such curves, the outer billiard map is continuous (although not everywhere differentiable). Outer billiards about piece-wise circular curves are relatively easy to study numerically.

1.4 Billiards in near squares

In the study of mathematical billiards, we fix a region enclosed by a curve in the plane. We consider a pointmass moving around a frictionless table and making elastic collisions with the boundary, so that the speed of the pointmass never changes. The pointmass bounces off the boundary so that the angle of incidence made with the tangent line equals the angle of reflection.

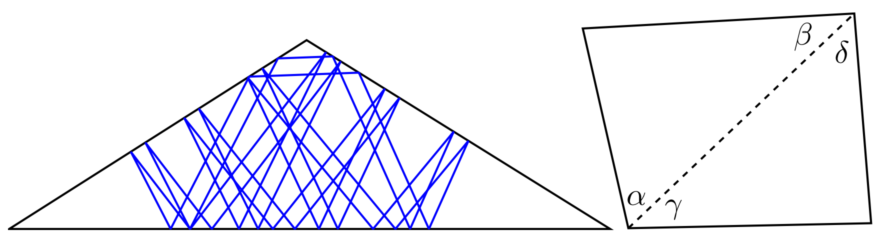

Figure 4: A periodic billiard path in a polygon, and an ϵ-near square.

It is natural to consider mathematical billiards in a polygonal region. See chapter 7 of [22] for an introduction to the topic. An open question is "does every polygonal billiard table admit a periodic billiard path?" Because this is a very difficult question, it is natural to try consider special cases of the question. Various special cases involving triangles have been considered. See [19] for a survey of some results for triangles.

It seems that billiards in quadrilaterals which are nearly squares would be an interesting special case. For $\epsilon > 0$, a quadrilateral is an *ϵ-near square* if the four angles made with a diagonal are within ϵ of $\frac{\pi}{4}$. *Is there an $\epsilon > 0$ so that every ϵ-near square has a periodic billiard path?* It is reasonable to expect that the answer to this question is interesting, and might be similar to the answer for the 30-60-90 triangle found in [18].

1.5 Spherical and hyperbolic versions of Gutkin's theorem

E. Gutkin asked the following question: given a plane oval γ, assume that two points, x and y, can "chase" each other around γ in such a way that the angle made by the chord xy with γ at both end points has a constant value, say, α. If γ is not a circle, what are possible values of α?

The answer is as follows: a necessary and sufficient condition is that there exists $n \geq 2$ such that $n \tan \alpha = \tan(n\alpha)$. See [21] for a brief proof.

In terms of billiards, the billiard ball map in γ has an invariant circle given by the condition that the angle made by the trajectories with the boundary of the table is equal to α. The result can be also interpreted in terms of capillary floating with zero gravity in neutral equilibrium, see [11, 12].

Problem: *find analogs of this result in the spherical and hyperbolic geometries. What about curves in higher dimensional spaces?*

1.6 A non-conventional billiard

Consider the following non-conventional "billiard". Let γ be an oval (the boundary of the billiard table). Let AB be the incoming trajectory, where $A, B \in \gamma$. The outgoing (reflected) trajectory is defined to be BC, $C \in \gamma$ where AC is parallel to the tangent line to γ at point B. See Figure 5.

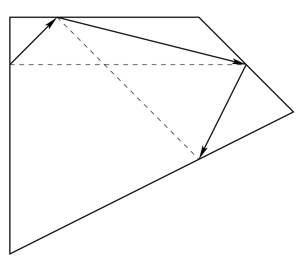

Figure 5: Non-conventional billiard

For example, periodic trajectories in this billiard correspond to inscribed polygons with extremal area (for comparison, periodic trajectories in the usual billiard correspond to inscribed polygons with extremal perimeter, and the ones in the outer billiards to circumscribed polygons with extremal area).

The general problem is to study these non-conventional billiards. In particular, what can be said if the billiard table is a polygon? The case of triangle is trivial, but quadrilaterals are already interesting.

2 Geometry

2.1 Origami hyperbolic paraboloid

There is a common origami construction depicted in Figure 6. The pleated surface looks like a hyperbolic paraboloid and is often called so in the origami literature.

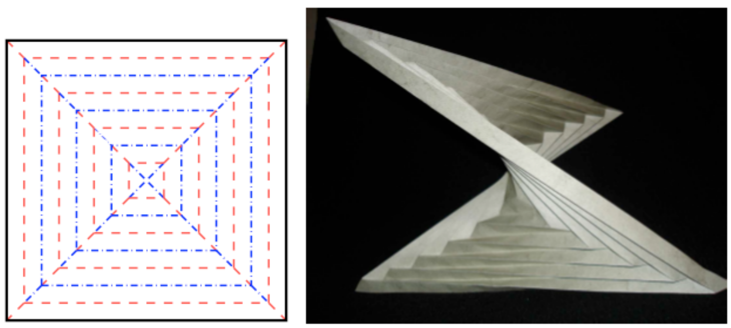

Figure 6: Hyperbolic paraboloid

The problem is to explain this construction. If one assumes that paper is not stretchable and the fold lines are straight then one can prove that this construction is mathematically impossible, see [7]. The explanation is that there exist invisible folds along the diagonals of the elementary trapezoids in Figure 6 left. Assuming this to hold, and given a particular patterns of these diagonals (one can choose one of the two for each trapezoid), *what is the shape of the piece-wise linear surface obtained by folding?*

A more general question: *what is the result of a similar construction for other patterns of folding lines?* See, e.g., Figure 7. See [13] concerning folding paper along curved lines.

2.2 The unicycle problem

A mathematical model of a bicycle is an oriented unit segment AB in the plane that can move in such a way that the trajectory of the rear end A is

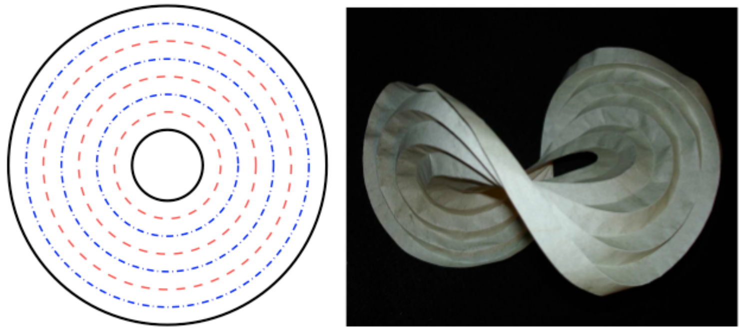

Figure 7: A different pleated surface

always tangent to the segment. Sometimes the trajectories of points A and B coincide (say, riding along a straight line).

The following construction is due to D. Finn [10]. Let $\gamma(t),\ t \in [0, L]$ be an arc length parameterized smooth curve in the plane which coincides with all derivatives, for $t = 0$ and $t = L$, with the x-axis at points $(0, 0)$ and $(1, 0)$, respectively. One uses γ as a "seed" trajectory of the rear wheel of a bicycle. Then the new curve $\Gamma = T(\gamma) = \gamma + \gamma'$ is also tangent to the horizontal axis with all derivatives at its end points $(1, 0)$ and $(2, 0)$. One can iterate this procedure yielding a smooth infinite forward bicycle trajectory $\mathcal{T}$ such that the tracks of the rear and the front wheels coincide. See Figure 8.

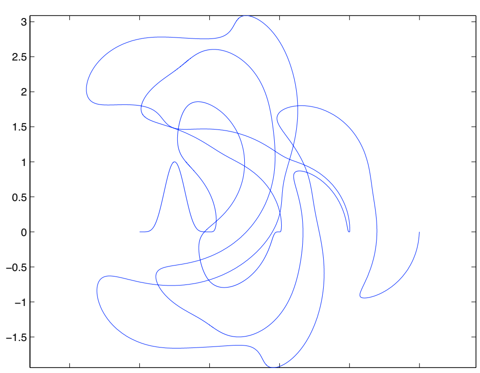

Figure 8: A unicycle track

It is proved in [17] that the number of intersections of each next arc of

$\mathcal{T}$ with the x-axis is greater than that of the previous one. Likewise, the number of local maxima and minima of the height function y increases with each step of the construction.

Conjecture: *Unless γ is a straight segment, the amplitude of the curve $\mathcal{T}$ is unbounded, i.e., $\mathcal{T}$ is not contained in any horizontal strip, and $\mathcal{T}$ is not embedded, that is, it starts to intersect itself.*

It is interesting to consider a piece-wise circular version of the same problem; it might be easier to tackle. See Figure 9.

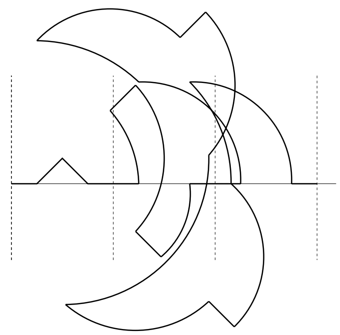

Figure 9: A piece-wise circular unicycle track

2.3 Geometry of bicycle curves and bicycle polygons

A closed curve γ in a Riemannian manifold M is called *bicycle* if two points x and y can traverse γ in such a way that the arc length xy remains constant and so does the distance between x and y in M. For example, a circle in Euclidean plane is a bicycle curve.

An explanation of the terminology is as follows. Let γ be a bicycle curve in the plane and let γ' be the envelope of the segments xy. If γ and γ' are the front and rear bicycle wheel tracks, then one cannot determine the direction in which the bicycle went from these curves. See [23] for a detailed discussion and references. Surprisingly, a bicycle curve can be also characterized as the section of a homogeneous cylinder (a log) that float in equilibrium in all positions.

Call the ratio of the arc length xy to the total arc length of γ the *rotation number* and denote it by ρ. In the plane, one can prove that, for some values of ρ (for example, $\rho = 1/3$ and $1/4$), the only bicycle curve is a circle, whereas for some other values (for example, $\rho = 1/2$), there exist non-circular bicycle curves; see [4, 23, 26, 27, 28]. A complete description of plane bicycle curves is not known. See Figure 10 for some examples (due to F. Wegner).

Problem: *construct non-trivial examples of bicycle curves on the sphere, the hyperbolic plane, multi-dimensional Euclidean space, etc.*

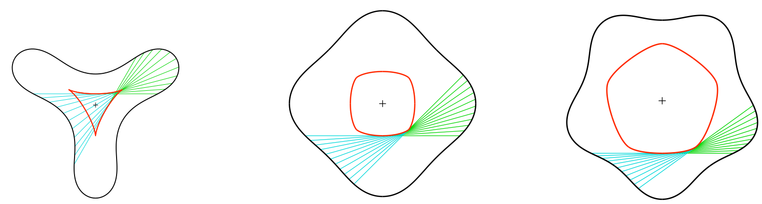

Figure 10: Non-trivial bicycle curves

A discrete analog of a bicycle curve is a bicycle (n,k)-gon, an equilateral n-gon whose k-diagonals are all equal to each other. For some values of (n,k) such a polygon is necessarily regular and for other values (say, $n=8, k=3$) non-trivial examples exist. A complete description is not known either; see [5, 6, 23] for partial results.

3 Algebraic geometry

3.1 Cayley-style theorem for null geodesics on an ellipsoid in Minkowski space

The following Poncelet-style theorem was proved in [14]. Consider an ellipsoid

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1, \quad a,b,c>0$$

in three dimensional Minkowski space with the metric $dx^2+dy^2-dz^2$. The induced metric on the ellipsoid degenerates along the two "tropics"

$$z=\pm c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}};$$

the metric is Lorentz, of signature $(+,-)$, in the "equatorial belt" bounded by the tropics. Through every point of the equatorial belt there pass two null geodesics of the Lorentz metric, the "right" and the "left" ones.

Call a chain of alternating left and right null geodesics, going from tropic to tropic, an (n, r)-chain if it closes up after n steps and making r turns around the equator. The theorem states that if there exists an (n, r)-chain of null geodesics then every chain of null geodesics is an (n, r)-chain. See [14] for a discussion.

Problem: *find conditions on the numbers a, b, c ensuring the existence of (n, r)-chains.* For the classical Poncelet porism, such a formula is due to Cayley, see [15]. For the Poncelet porism, see [3] and [1].

3.2 A converse Desargues theorem

A classical Desargues theorem states the following. Consider a pencil of conics (a one-parameter family of conics sharing four points – these points may be complex or multiple, as for the family of concentric circles). The intersections of a line ℓ with these conics define an involution on ℓ, and the theorem states that this involution is a projective transformation of ℓ. See [2].

Let $f(x, y)$ be a (non-homogeneous) polynomial with a non-singular value 0. Let γ be an oval which is a component of the algebraic curve $f(x, y) = 0$. Assume that the curves $\gamma_\varepsilon = \{f(x, y) = \varepsilon,\ \varepsilon > 0\}$ foliate an outer neighborhood of γ and that for every tangent line ℓ to γ, its intersections with the curves γ_ε define a (local) projective involution on ℓ. **Problem**: *prove that γ is an ellipse and the curves γ_ε form a pencil of conics.*

A particular case, in which the involutions under consideration are central symmetries of the line, is proved in [25].

3.3 New configuration theorems of projective geometry

A number of new configuration theorem of elementary projective geometry were discovered in [20], see, e.g., Figure 11. These theorems somewhat resemble the classical theorems of Pappus and Pascal. However only in some cases geometric proofs are known; the rest is a "brute force" computer computation (and in one case, the computation is too large for computer). The problem is to understand what is going on, to find conceptional proofs and possible generalizations to these theorems. (In particular, the last theorem in [20] is not really a theorem, since one only has a numerical "proof" for it).

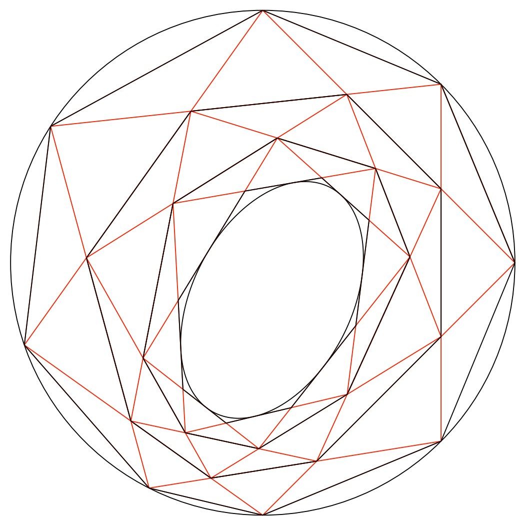

Figure 11: A configuration theorem: the inner-most octagon is projectively dual to the outer-most one

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