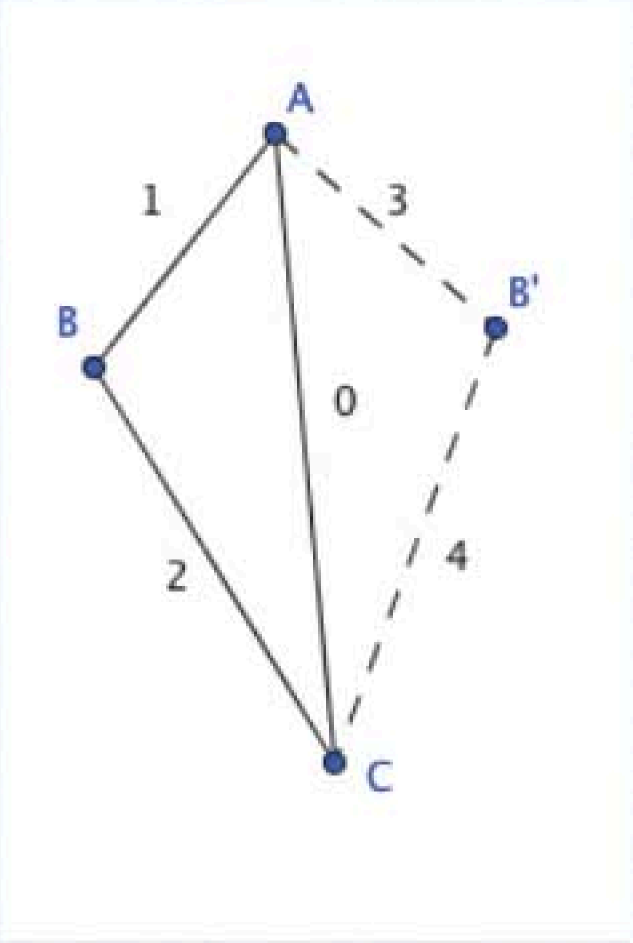
A cute little Rhombus

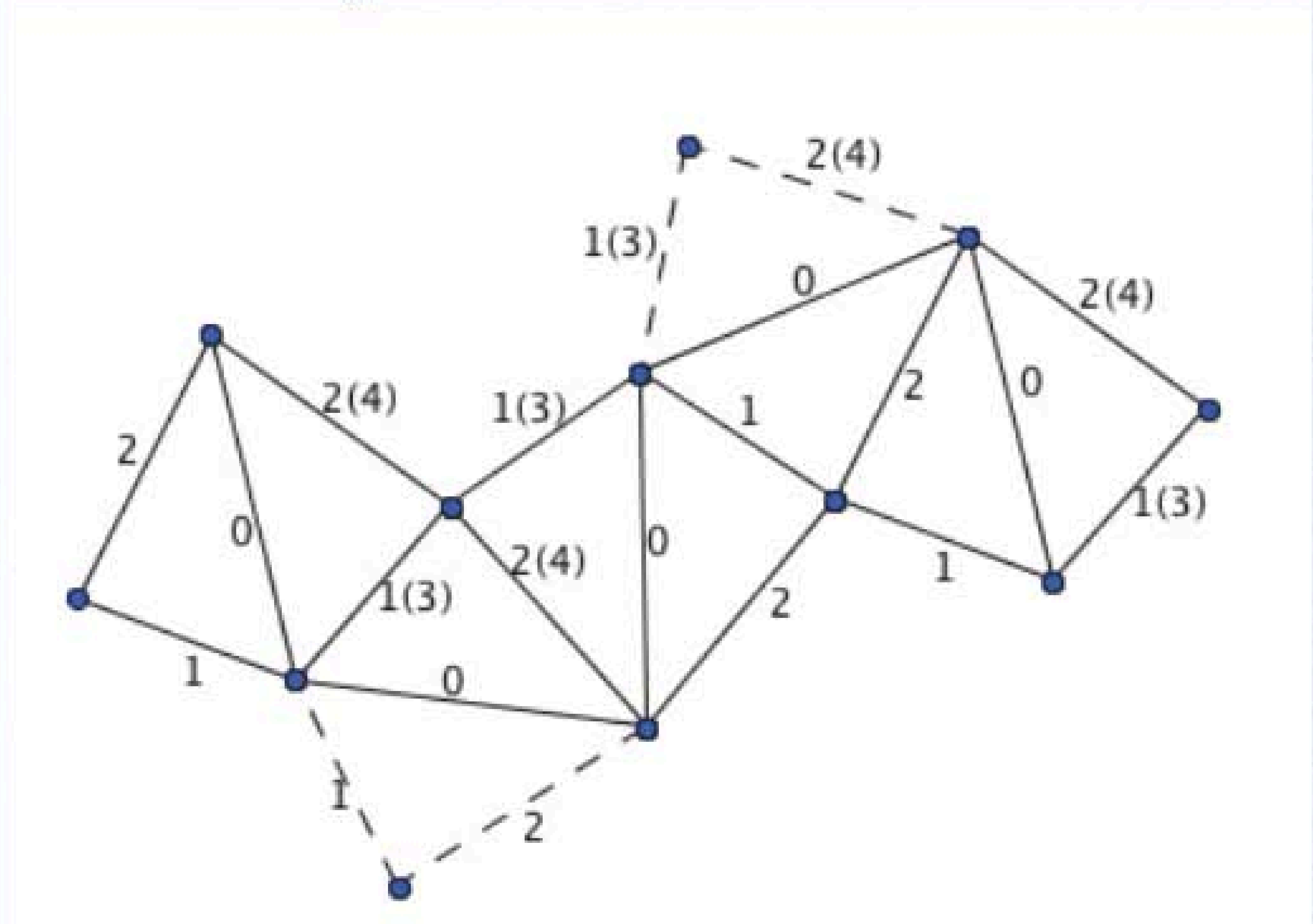
Billiards in triangles and in kites

- Given any triangle, we can reflect it into a kite.
- We may label the three edges of the triangle 0,1,2 respectively. Then we reflect it along 0. We shall label the image of edge 1 and 2 as 3 and 4 respectively. Then we have a kite with edges labeled 1,2,3 and 4.



Billiards in Triangles and in Kites

- For any periodic billiard path in a triangle, we can always complete it into a periodic billiard path for the corresponding kite.
- For any periodic billiard path in a kite, in the unfolding, we can split each kite into two corresponding triangles, then the result is a periodic billiard path for the triangle.

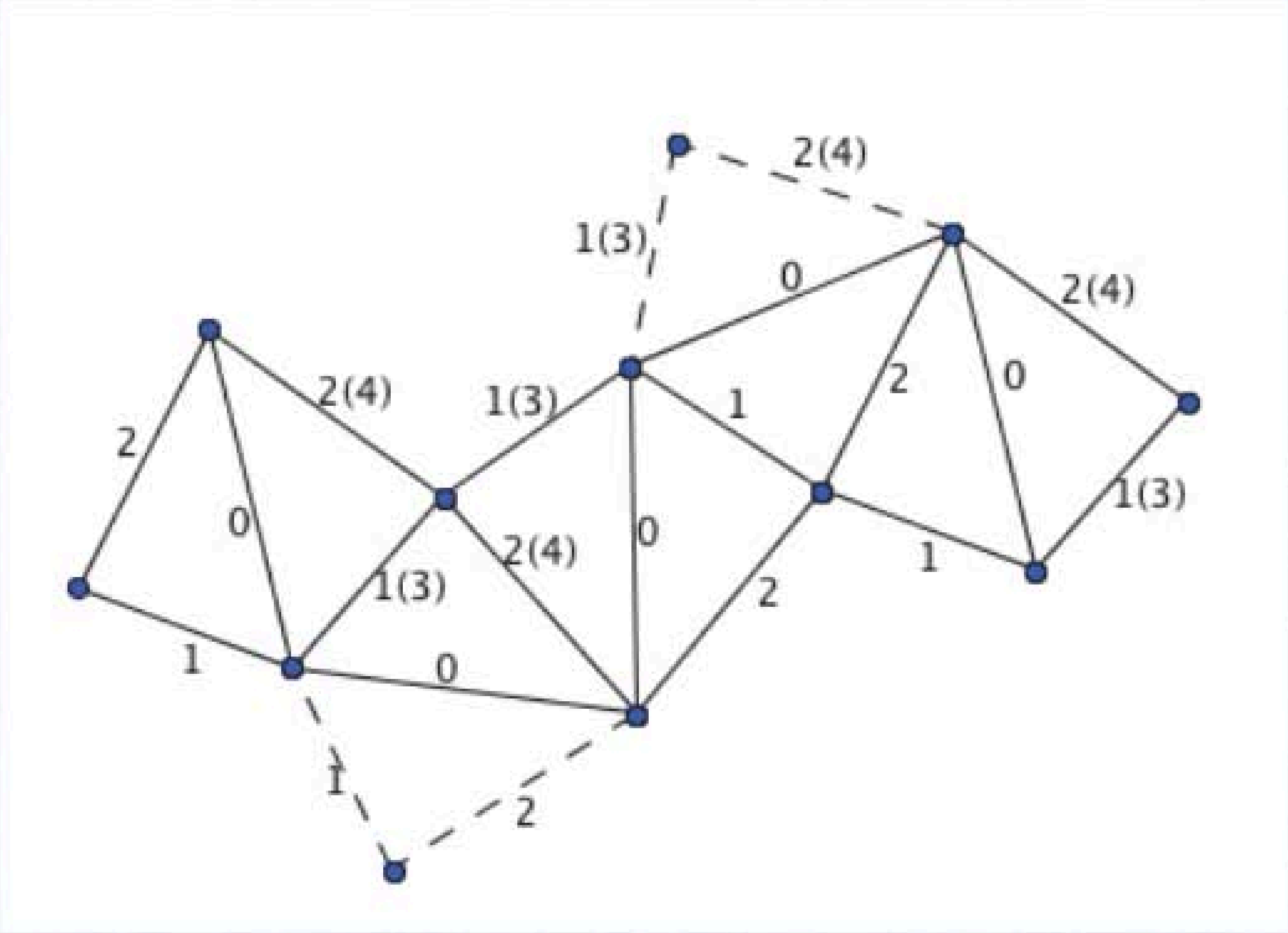


Billiards in Triangles and in Kites

- How to change from a triangle billiard path to a kite billiard path:
  - First of all, all 0-edges will disappear in the orbit type.
  - Whenever you hit a 0-edge, from there on 1-edges and 2-edges will change to 3-edges and 4-edges respectively, or 3-edges and 4-edges will change to 1-edges and 1-edges respectively

Billiards in triangles and in kites

- Say triangle orbit type 012012. Changing to kite orbit type, it becomes 3412.



Billiards in Triangles and in Kites

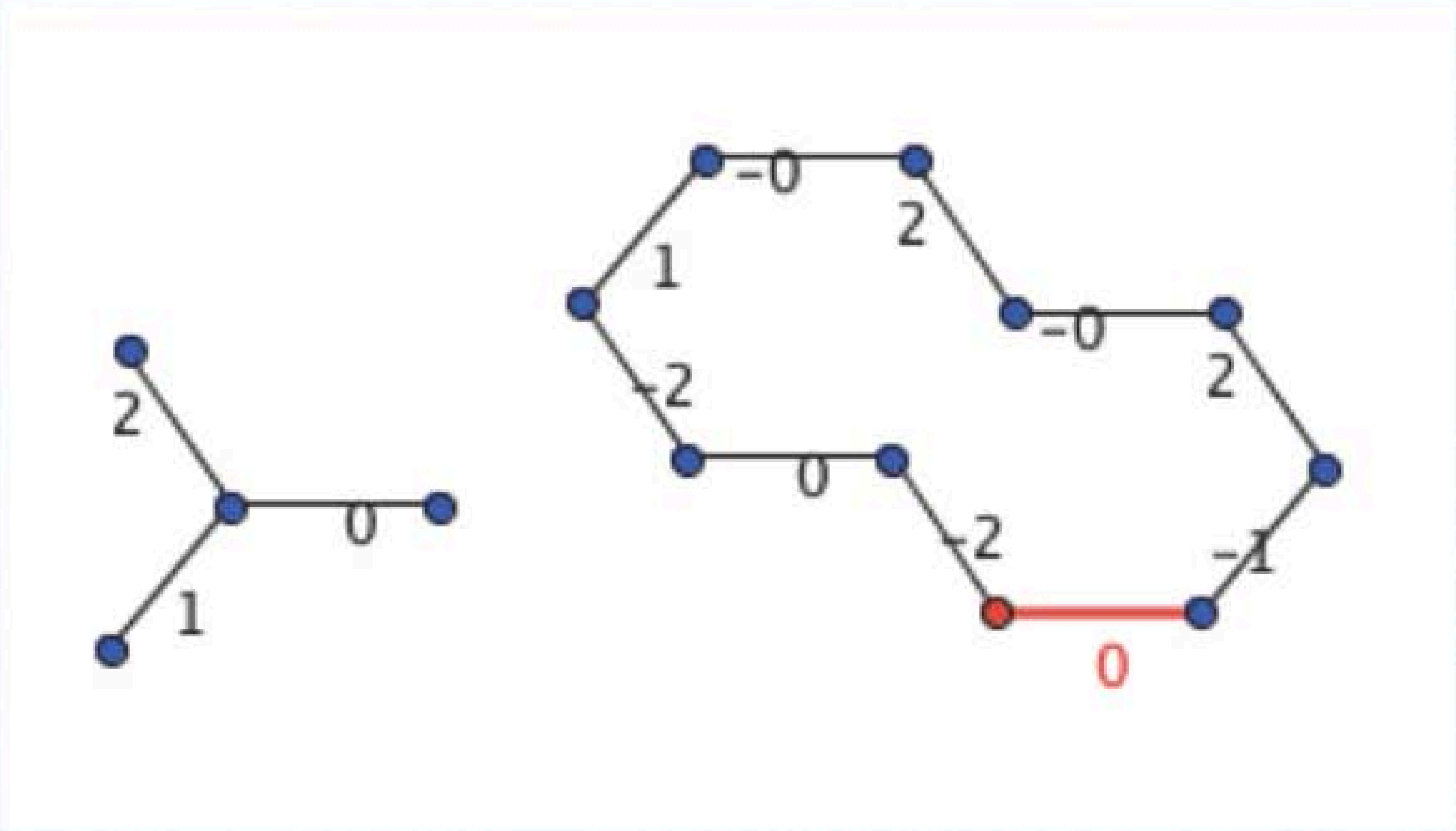
- All stable kite orbits corresponds to stable triangle orbits. The converse is NOT true.
- So we can use McBilliards to search stable orbits for triangles, and then use the above algorithm to check stability of the corresponding orbit for the corresponding kite.
- Rhombi are special kind of kites corresponding to isosceles triangles.

Hexapath for Triangles

- Pick any three nice enough vectors in R^2. Label them 0,1,2.
- For any orbit type for triangles, say "012012", we can move according to the following laws:
  - We start from a point, and move according to digit in the orbit type one by one.
  - When we see 0,1,2 in the odd position, then we move by the corresponding vectors.
  - When we see 0,1,2 in the even position, then we move by the negation of the corresponding vectors.

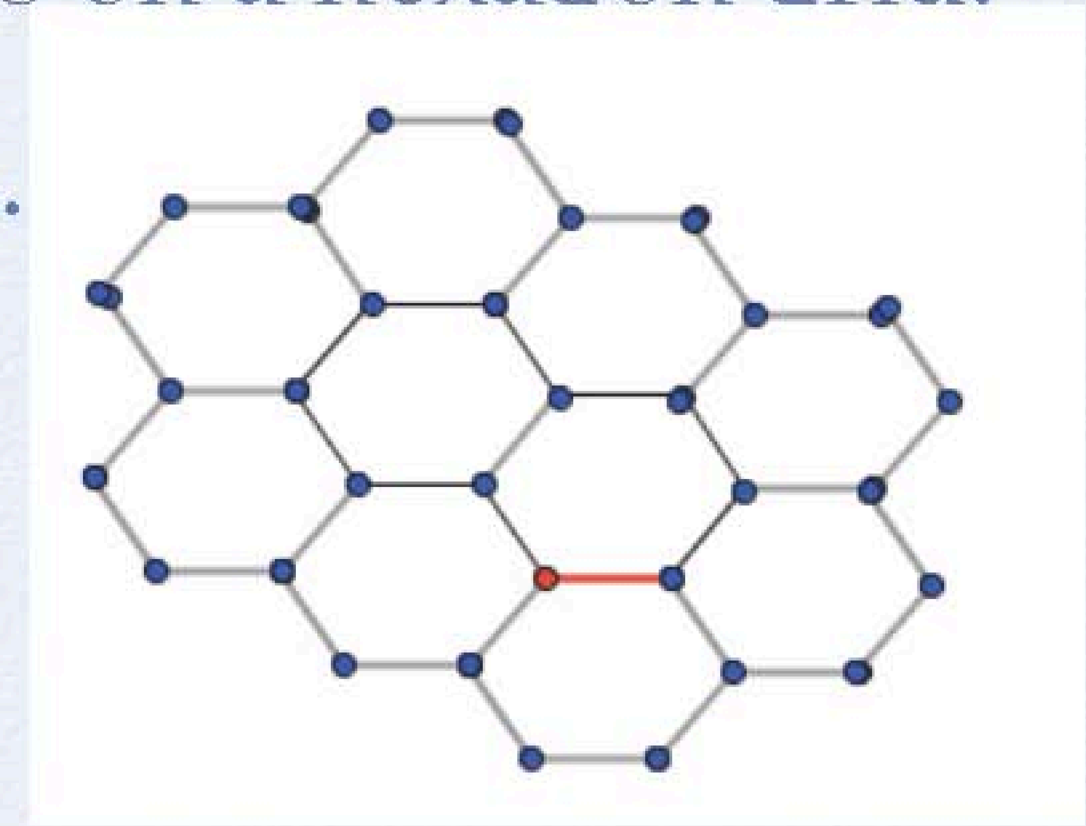
HEXAPATH FOR TRIANGLES

- The hexapath for orbit 0120201202:



Hexapath for Triangles

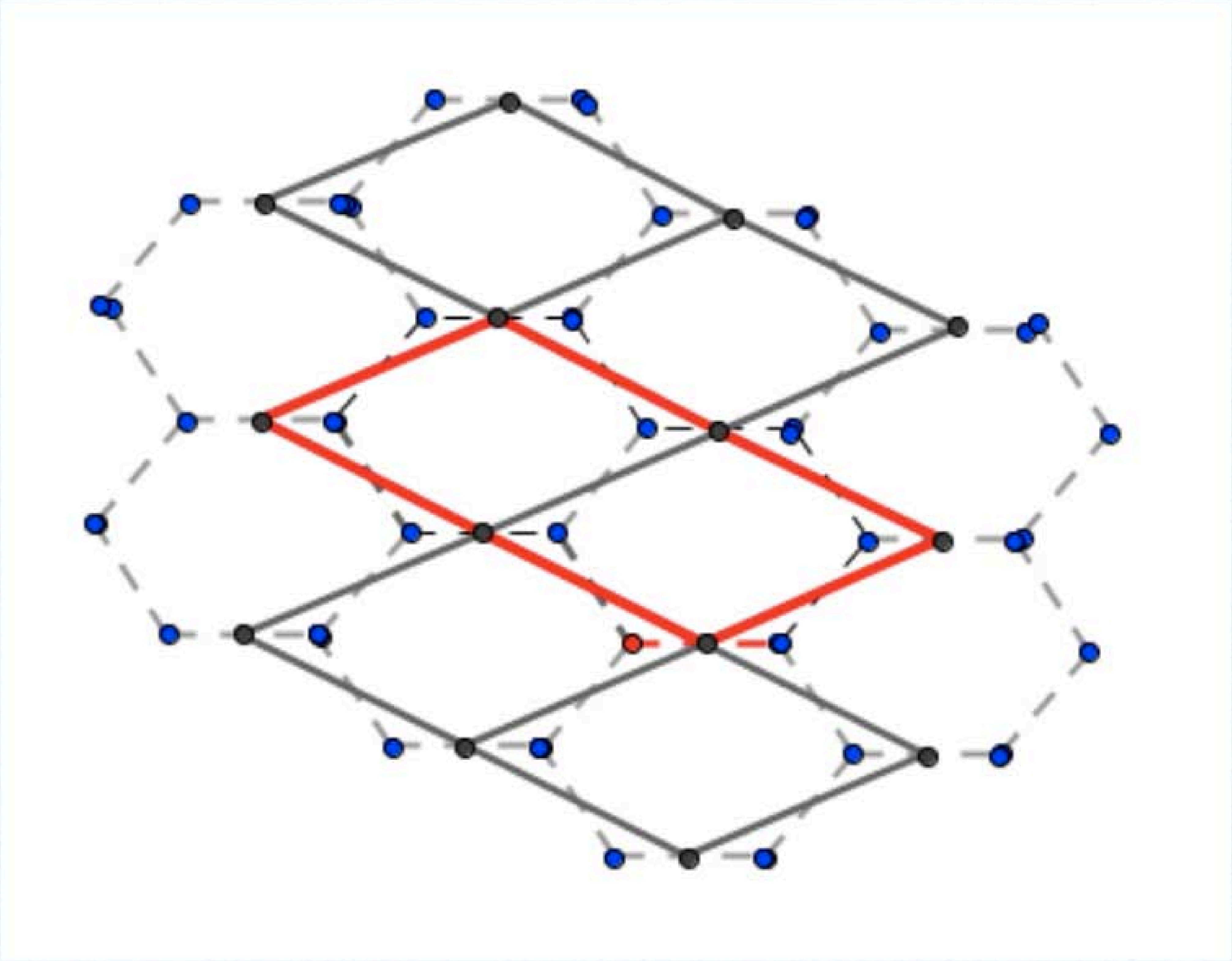
- Nice things about hexapaths:
  - An orbit type is stable iff the hexapath is a closed loop.
  - The path always stays on a hexagon grid.
  - Easier to see patterns.



Square path for triangles

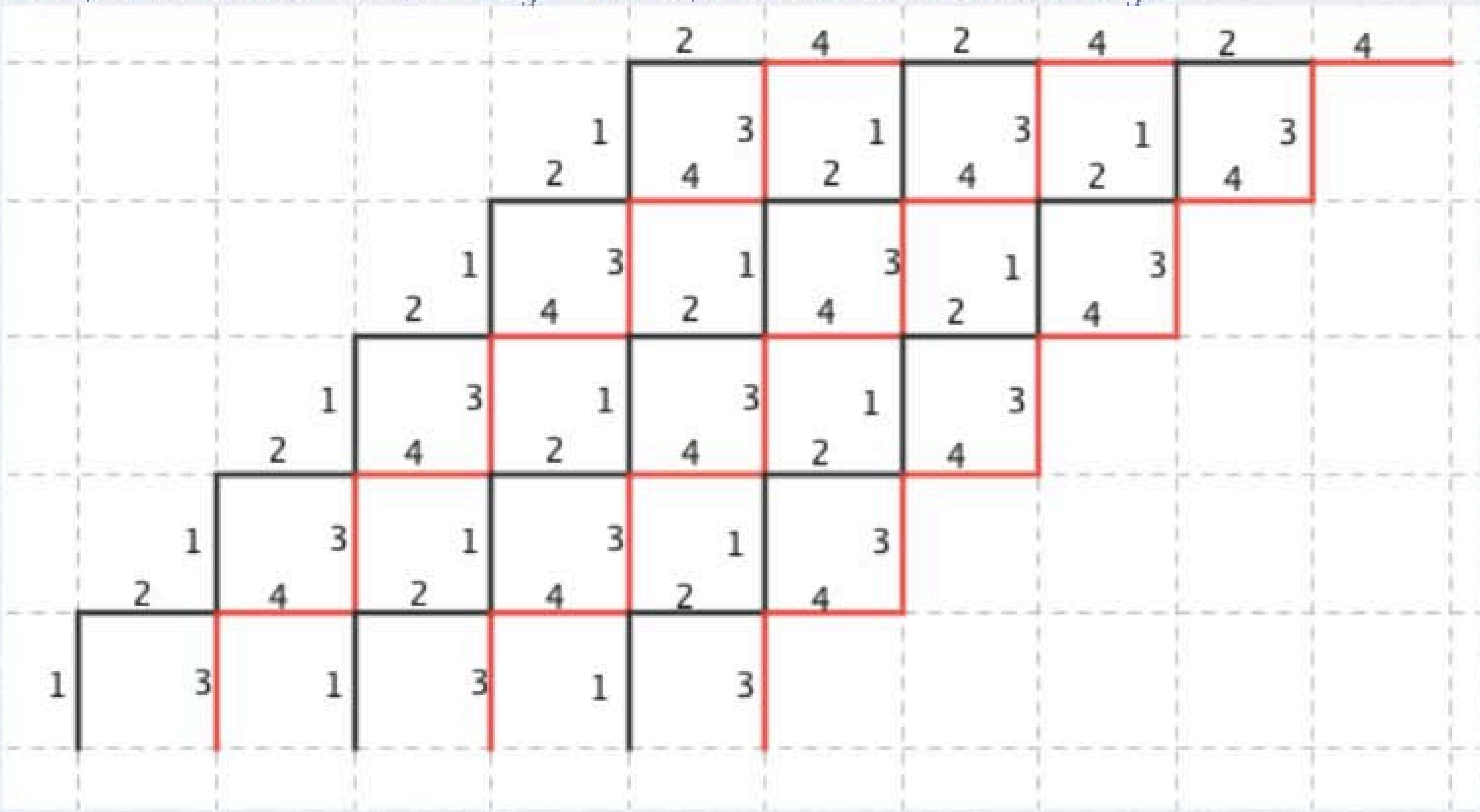
- Consider the hexagon grid:
  - On each grid point there is a unique edge with label 0.
  - Connect the center of these edges gives us a square grid
  - on the square grid we can find the square path.

Square Path for Triangles



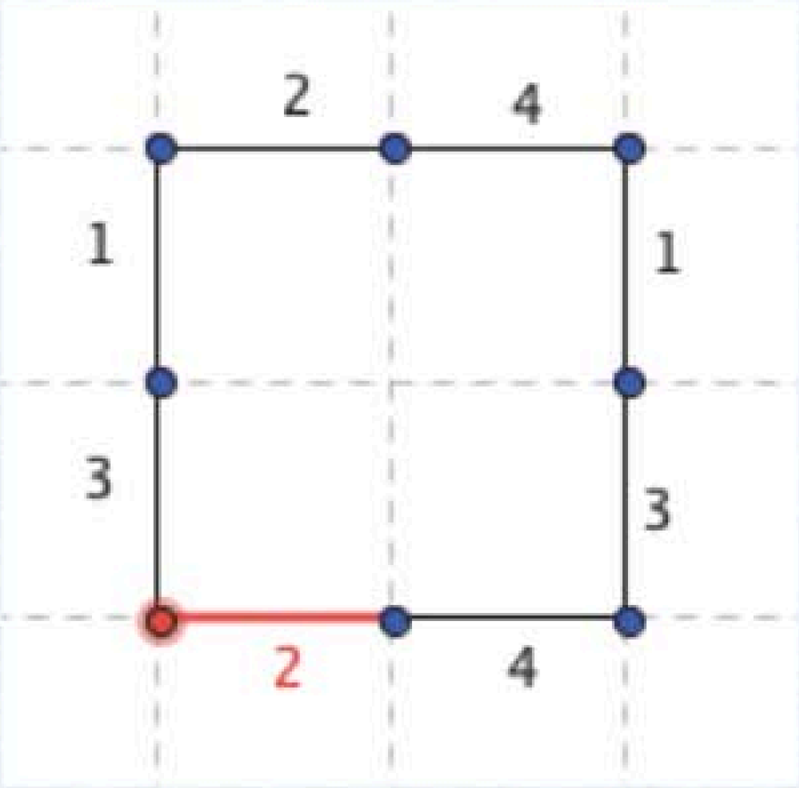
Square Path for Quadrilaterals

- We start by a square grid. Then we break the square grid down into “staircases” . On each staircase we label 1, 2 alternatively or 3, 4 alternatively.



Square Path for Quadrilaterals

- Clearly for each grid point, there is a unique edge with each of the four label.
- Given an orbit type, say 42312413, we start from an arbitrary point and draw the square path.
-



Square Path for Kites

- Given an orbit type for kites, we can draw their quadrilateral square path.
- We can also consider it as the orbit for the corresponding triangle. Then we can draw the triangle square path.
- The two square path always coincide.

π/N-RHOMBI

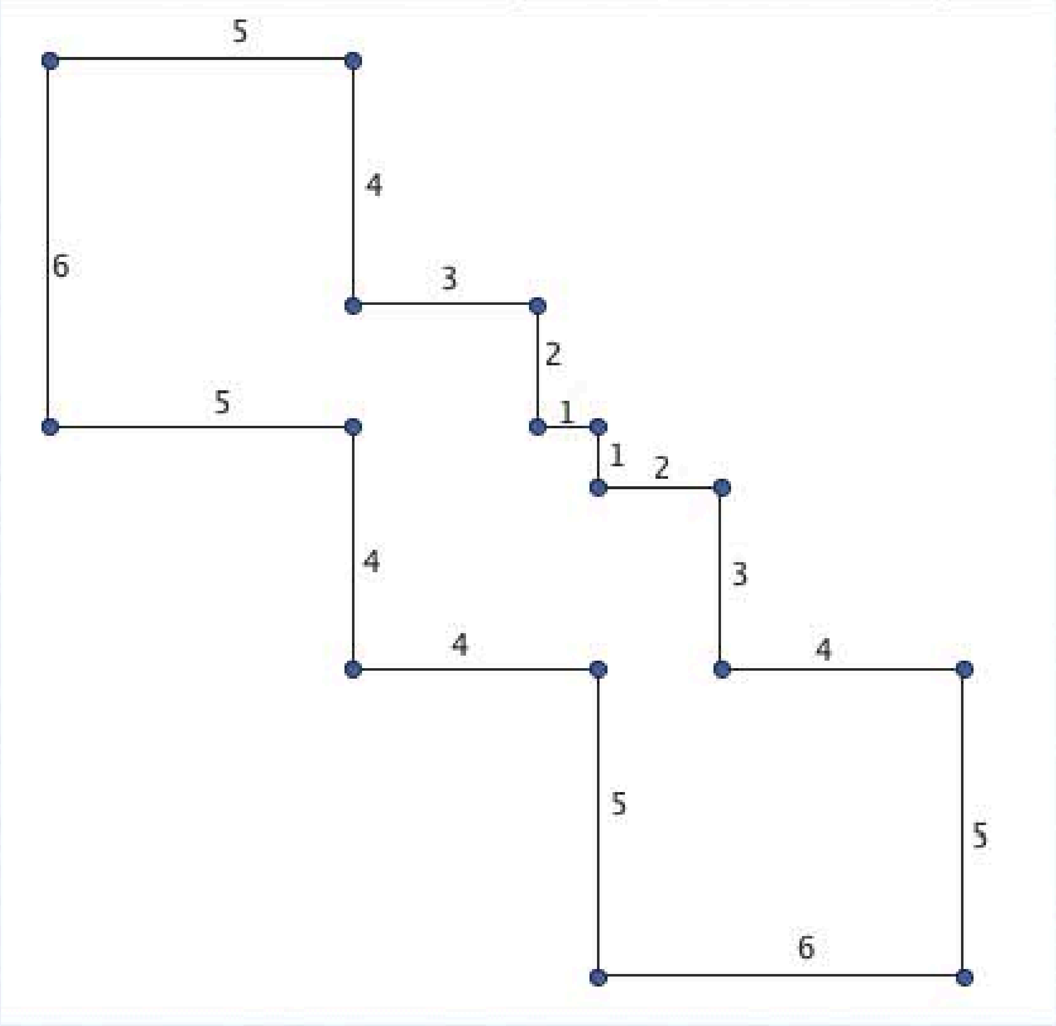
- A -isosceles triangle refers to an isosceles triangle whose acute angle is .
- All isosceles triangles have stable periodic billiard path, except for $\pi/(2^n)$-isosceles triangles.(Sch & Hoo 2009)
- A -rhombus refers to a rhombus whose acute angle is .
- Conjecture: All rhombi have stable periodic billiard path, except for $\pi/(2^n)$-rhombi.
- Sadly, almost all orbits used in Sch & Hoo’s paper are no longer stable for the rhombi.

π/N-RHOMBI

- Theorem: All $\pi/$ n-rhombi have stable periodic billiard path, except for $\pi/(2^n)$-rhombi.
- We shall show that if n has a odd factor, then $\pi/$ n-rhombi have stable periodic billiard path.

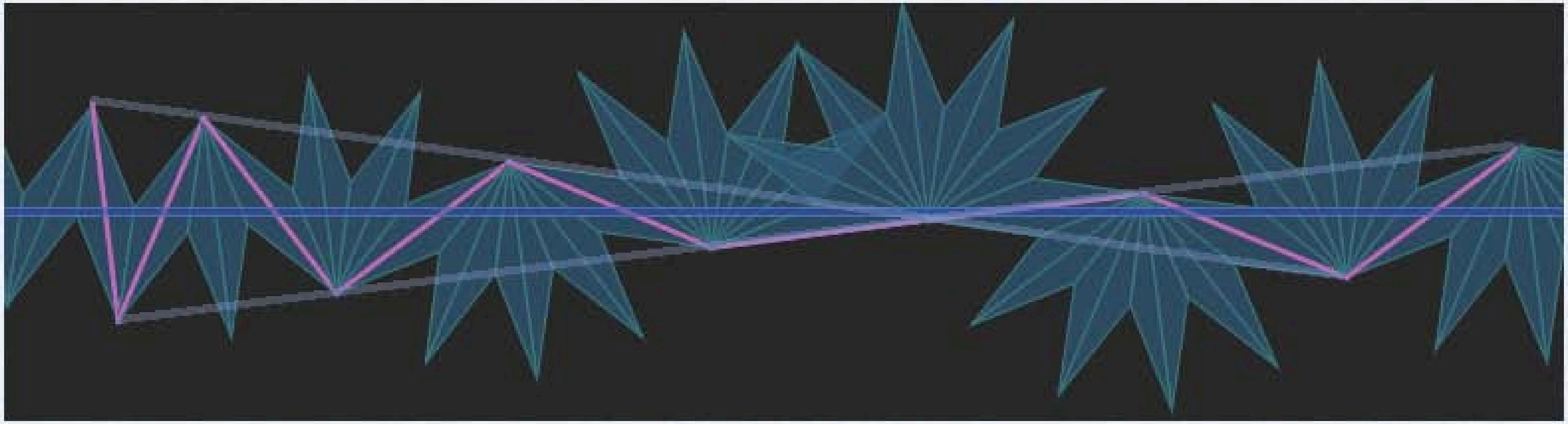
π/N-RHOMBI

- Proposition: If n is the sum of consecutive positive integers, i.e. $n=a+(a-1)+(a-2)+...+(a-k)$, and n and a have different parity, then π/n-rhombi have stable periodic billiard path. (e.g. 5=2+3 is bad)
- Square path of the orbit for $\pi/6$-rhombi: (6=1+2+3)



π/N-RHOMBI

- Proof of the proposition:
  - Analyzing the two gray lines + calculation.



π/N-RHOMBI

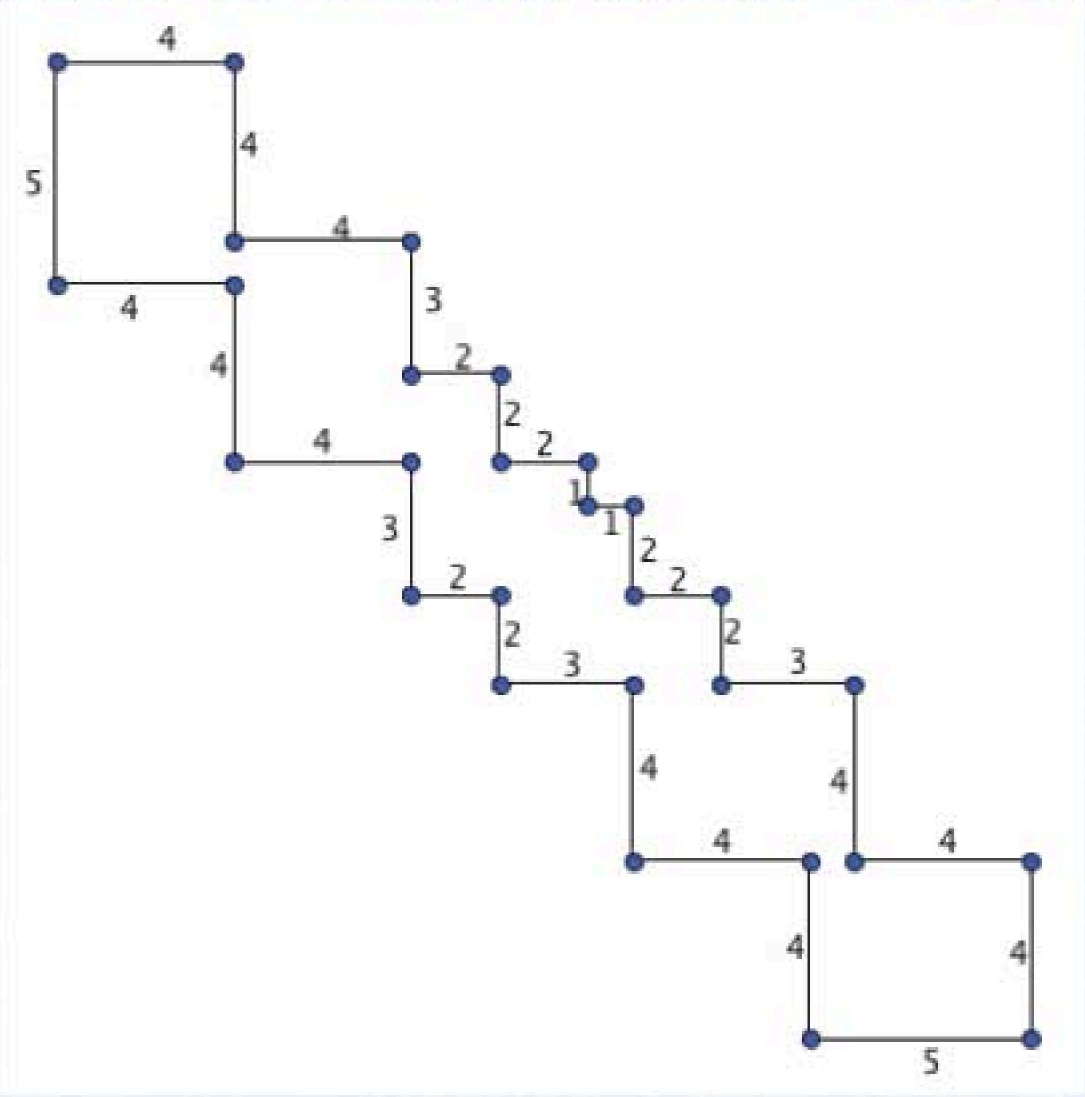
- Proposition: If n has a factor congruent to 3 mod 4, then it satisfies the condition in the previous proposition.
- Proof:
  - 7=4+3 =4+3+2+1+0+(-1)+(-2)
  - 14=5+4+3+2 =5+4+3+2+1+0+(-1)
  - 21=6+5+4+3+2+1 =6+5+4+3+2+1+0
  - 28=7+6+5+4+3+2+1 =7+6+5+4+3+2+1

π/N-RHOMBI

- Some fact about numbers again:
  - If n is congruent to 1 mod 4:
  - 5=2+2+1+0, 9=3+3+2+1, 13=4+4+3+2, ...
  - If n has a factor congruent to 1 mod 4:
  - 13=4+4+3+2+2+2+1+0+0+0+(-1)+(-2)+(-2)
  - 26=5+5+4+3+3+3+2+1+1+1+0+(-1)+(-1)

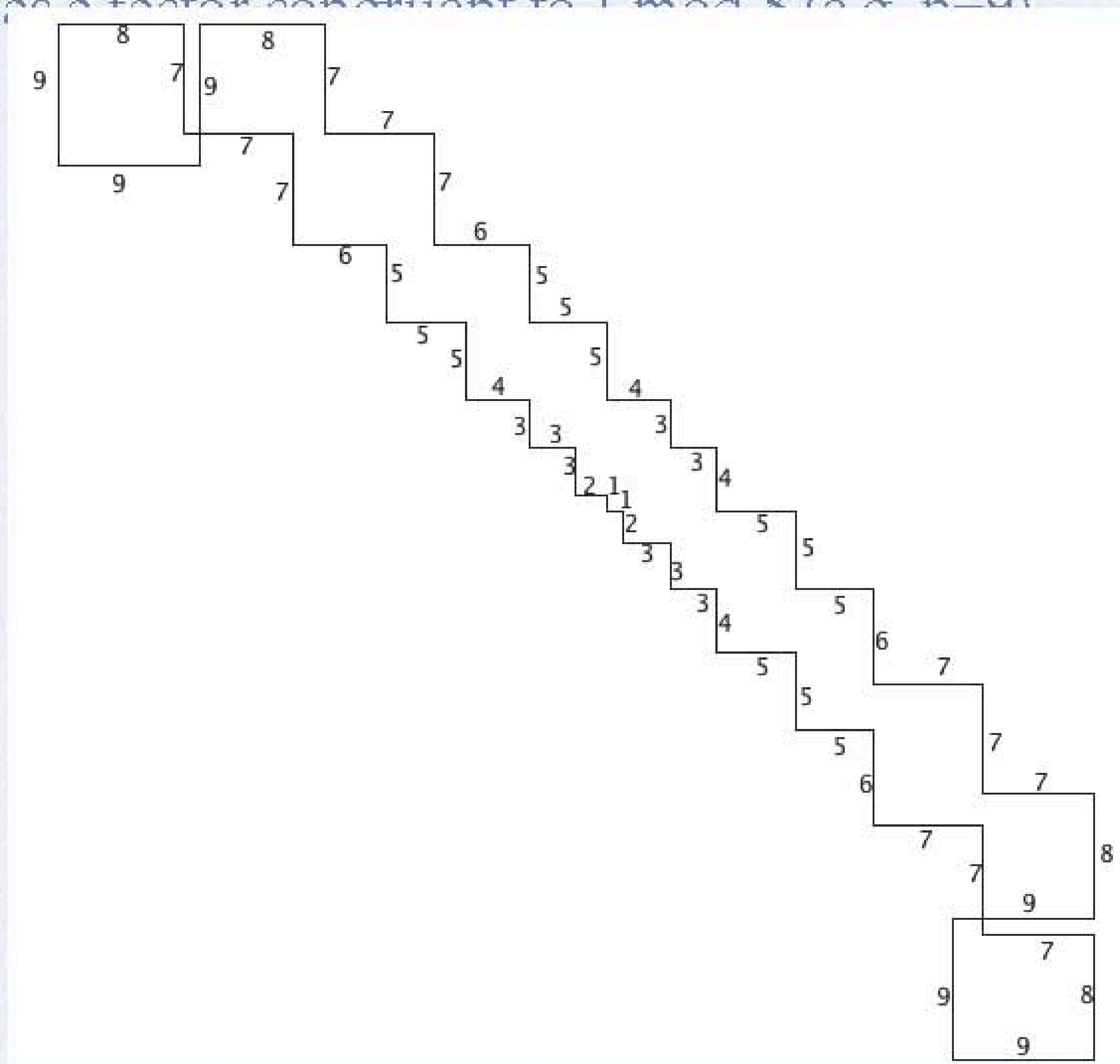
π/N-RHOMBI

- Square path for corresponding orbit:
  - n has a factor congruent to 5 mod 8



π/N-RHOMBI

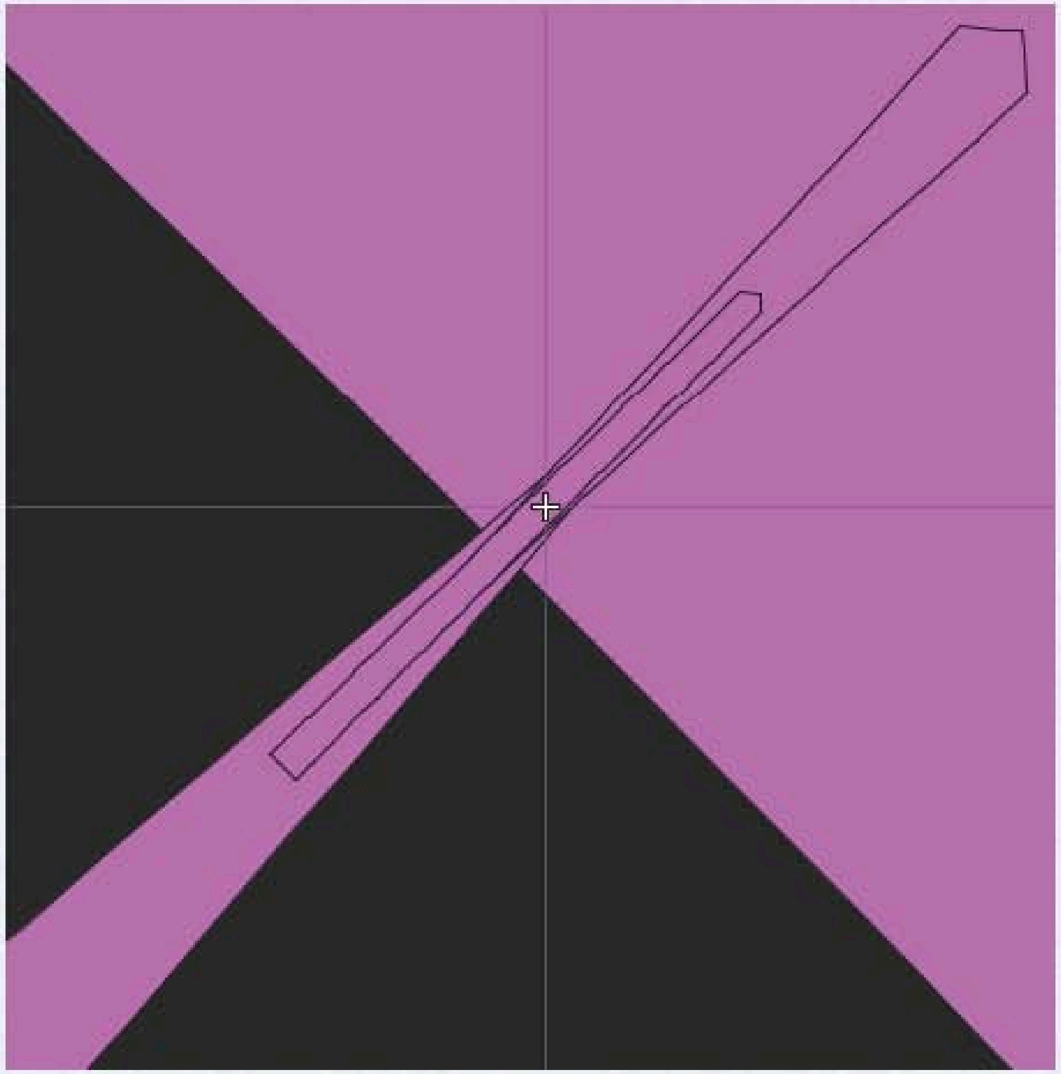
- n has a factor congruent to 1 mod 8 (e.g. n=9)



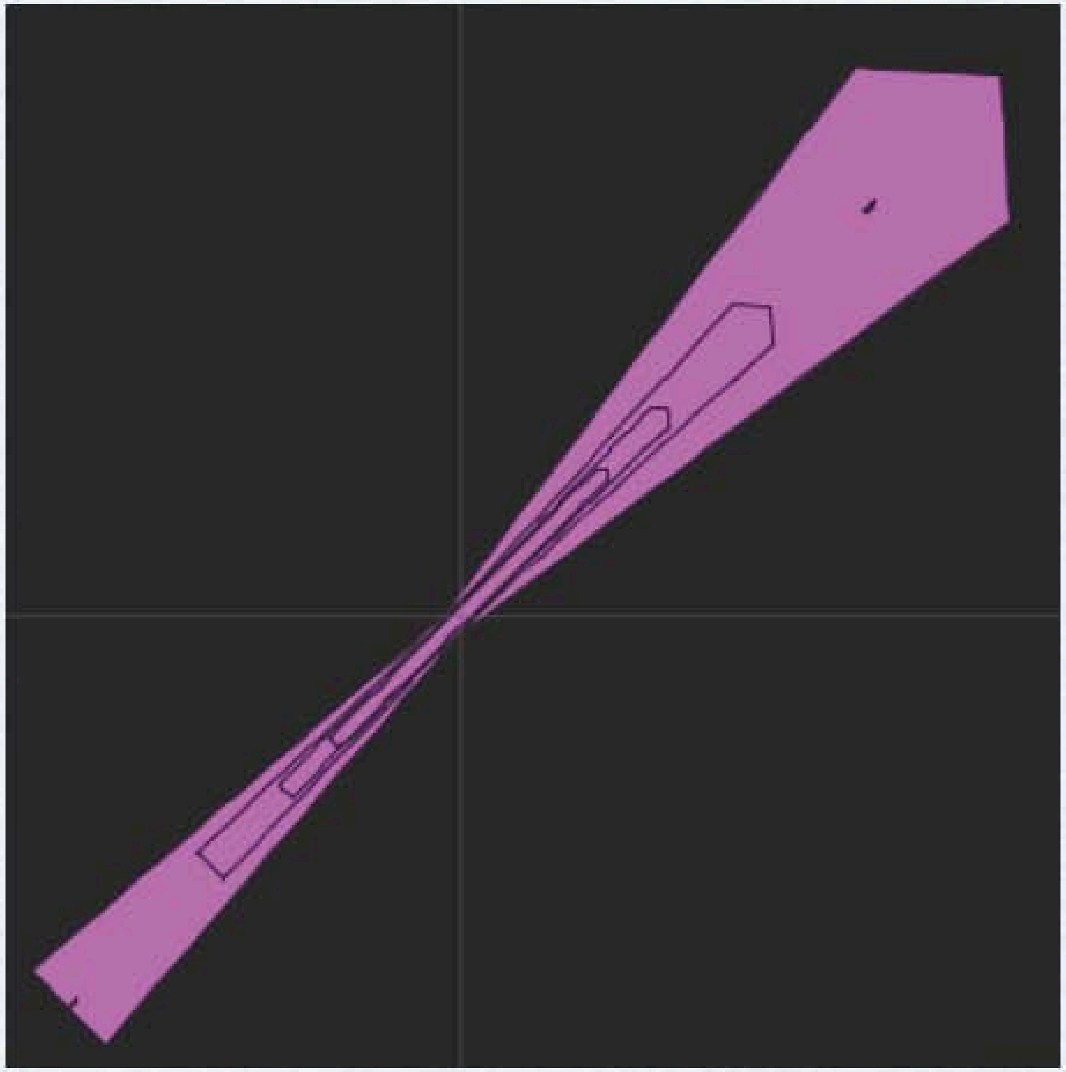
Butterflies

- Observation: For all odd n>3, all $2\pi/n$-rhombi have a double infinite family $A_{m,n}$ of stable periodic billiard paths.

n=4k+1

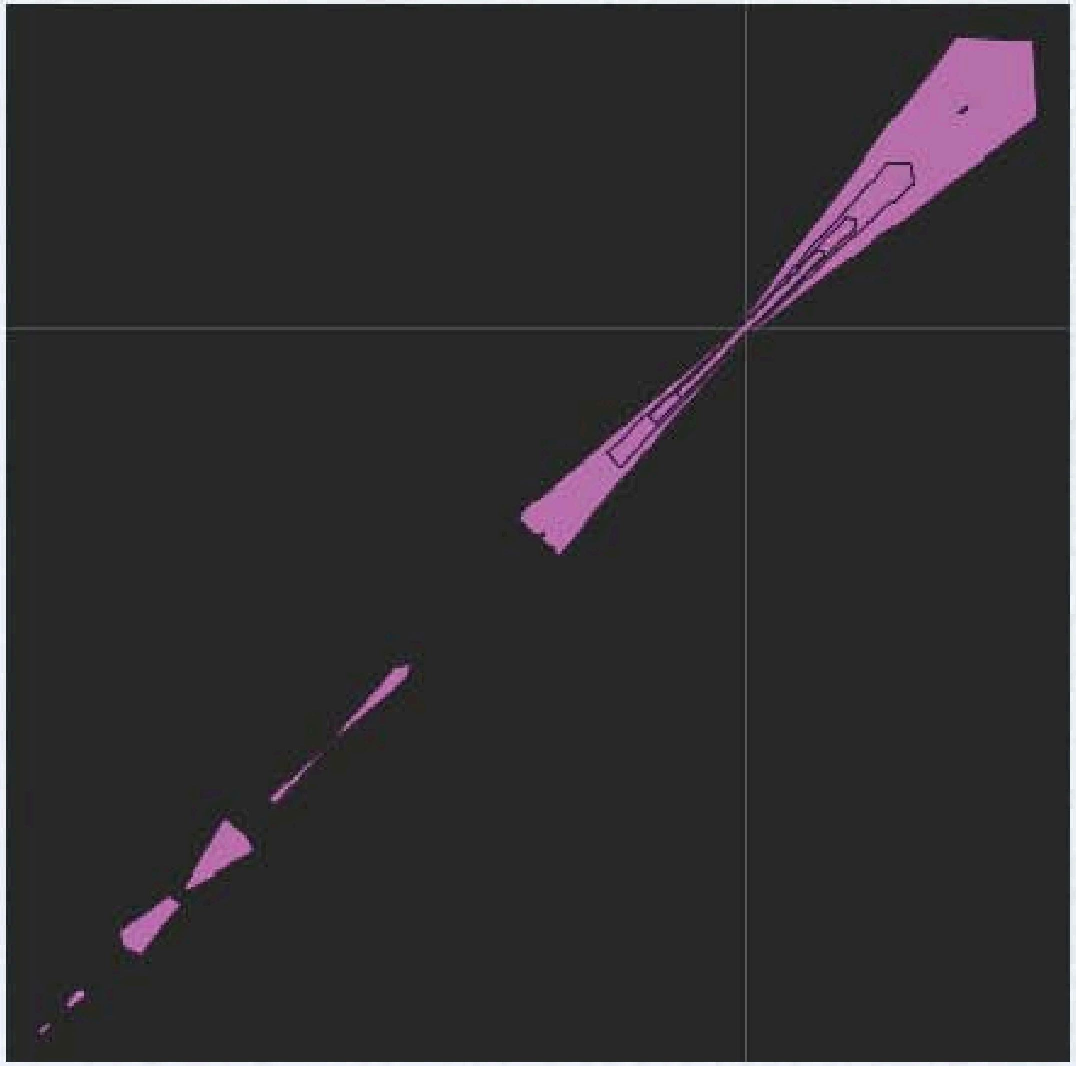


n=4k+3



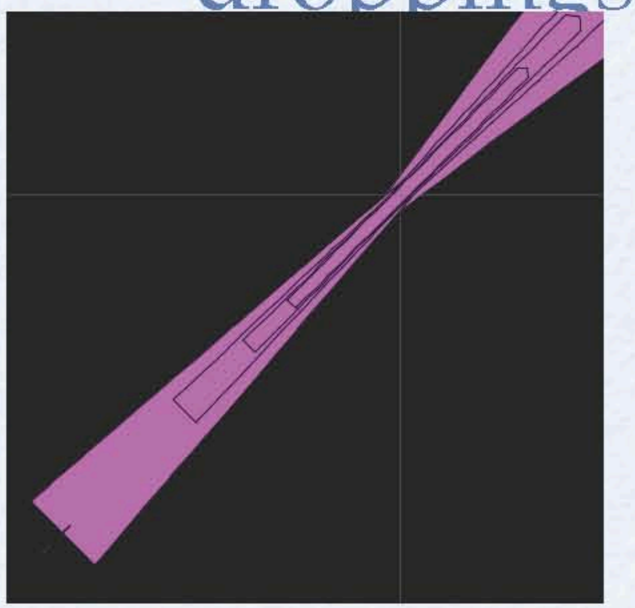
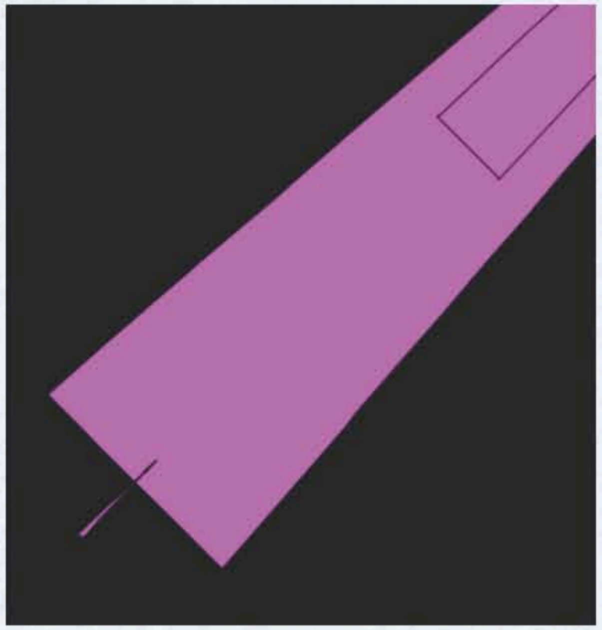
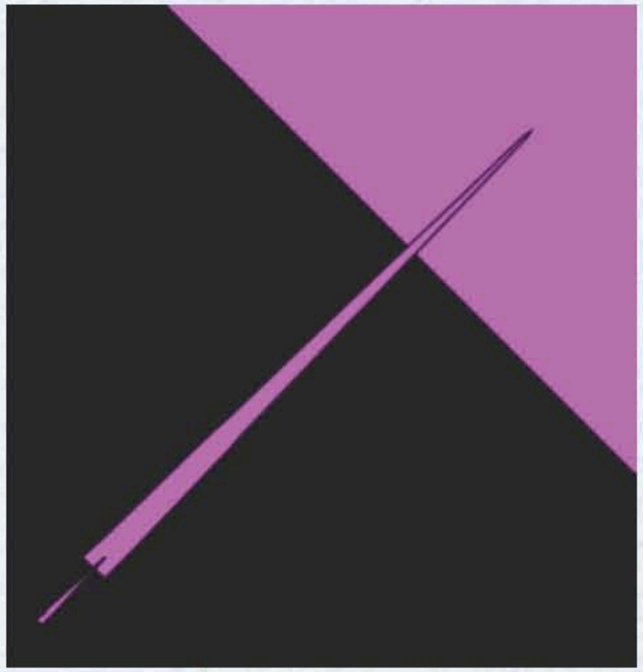
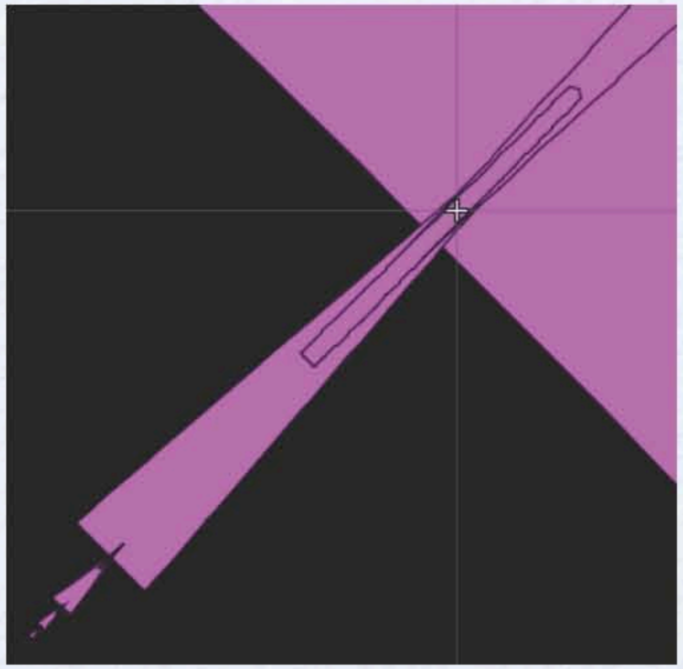
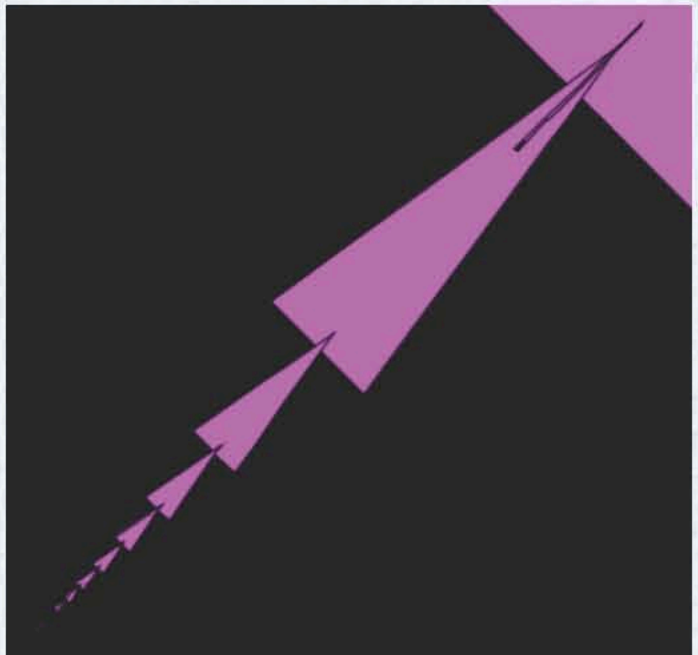
BUTTERFLIES

- Sadly, the case for $4\pi/n$ is much more complicated. (Especially $4\pi/(8k+5)$)
- But we do have some nice butterflies, each is a double family of stable orbits.



BUTTERFLIES

- And each butterfly have some "butterfly droppings" ....

Butterflies

- The square path pattern for the butterflies and droppings are super nice and easy to study. The square path are all simple closed loops with no self-intersection.
- The butterflies don't exist between $\pi/2$ and $\pi/4$, but are abundance between $2\pi/(4k+3)$ and $\pi/(2k+2)$, and between $\pi/(2k)$ and $3\pi/(6k+2)$
- However, the gaps between them are hard to fill up.

A cute little Rhombus

- Thank you!